OPTIMAL CONTROL OF THE M/G/1 QUEUEING SYSTEM WITH REMOVABLE
SERVER-LINEAR AND NON-LINEAR HOLDING COST FUNCTION

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Peter Orkenyi

In recent years, various queueing control problems have been studied
and solved by a number of investigators. A brief, but excellent survey
of the literature on this can be found in Gross and Harris (1974, pp. 364-
371). In most cases, the studies have concerned a single server. In
this report, we consider the M/G/1 queueing system with removable server.

In Section 1, the problem is defined, some potential applications
are outlined, and previous studies of the problem are reviewed. The
problem is then formulated as a semi-Markov decision process in Section
2. In Section 3, the case of linear holding cost is considered. Finally,
the case of non-linear holding cost is considered in Section 4.

1.   Introduction.

The M/G/1 queueing system with removable server was first studied
by Yadin and Naor (1963). Their idea was to utilize the idle time of
the server in the M/G/1 queueing system, since this time can be sub-
stantial. Therefore, they proposed to remove the server when the system
would become empty (thus letting the server perform some other useful
duty), and to bring him back when the number of customers in the system
would reach a certain critical number. We investigate this idea by con-
sidering the optimal control of the queueing system.

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1.1 Problem Formulation

The queueing system under consideration here has a single server with the customers arriving according to a stationary Poisson process with rate λ (> 0). They are served sequentially, and the service times are independent random variables with common cumulative distribution function F. Let μ be the service rate. It is assumed that the load on the system, $\rho = \lambda/\mu$, is less than one. The service times are assumed to be independent of the arrival process.

At each point in time, the server is either on or off. While off, no customers are being served. While on, the customers are served just as in the ordinary M/G/1 queueing system. An on server may be turned off at any point in time except when the server is giving service to a customer. An off server may be turned on by initiating a start-up procedure at the end of which the server is on. Its duration is the start-up time. It is assumed that the start-up times are independent random variables with common cumulative distribution function G. It is also assumed that the start-up times are independent of the arrival process and the service times.

The cost structure consists of four types of costs. First, there is a service cost K incurred each time a service is initiated. Second, there is an idling cost incurred at a rate r while the server is on but has no customers to serve. Third, there are switching costs. A start-up cost R_1 and a shut-down cost R_2 is incurred upon each completion of a start-up and a shut-down, respectively. Fourth, there is a holding cost for holding customers in the system. It is incurred at a rate which is a non-negative, non-decreasing function (h) of the number of customers in the system.

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Some comments about these costs are appropriate here. The service cost K may actually represent the expected (discounted) cost for giving service to a customer. Likewise, R_1 and R_2 may actually represent the effect of costs incurred during the start-up and shut-down times, respectively (although the shut-down times are not considered explicitly here, they are not excluded by our formulation of the problem). We assume that r, R_1 and $R_1 + R_2$ are non-negative.

In general terms, the objective is to find a policy for turning the server on and off such that expected costs are minimized. The problem is considered both with and without the use of discounting. When the costs are not discounted, two optimality criteria are used. The first one is the average cost criterion, according to which a policy is optimal if it minimizes the long run expected average cost. The second criterion is the undiscounted cost criterion. A policy is optimal for this criterion if it minimizes the long run expected cost where a cost incurred at a rate equal to the minimum long run expected average cost is subtracted from the original costs. When the costs are discounted, the discounted cost criterion is used. A policy is optimal for this criterion if it minimizes the total expected discounted cost.

We will let α denote the interest rate, N denote the set of positive integers, N_0 denote the set of non-negative integers, and R denote the set of real numbers.

1.2 Examples of Potential Applications.

Traffic Control:

Consider a bridge which can be opened and closed at a cost r_1 and r_2, respectively (for the sake of simplicity, we assume that they are

incurred at the completion of the operations). A ship can only pass under the bridge if it is open. Thus, if a ship arrives and the bridge is closed, the ship must wait until the bridge is opened before it can pass under it. There is a cost for keeping the ships waiting, and it is incurred at a rate h for each ship. The flow of traffic on the bridge is stopped when the bridge is opened, and it only resumes when the bridge is closed again. A cost is incurred at a rate r when the traffic on the bridge is interrupted. The problem is to determine when the bridge should be opened and closed.

That this problem can be viewed as an $M/G/1$ queueing system with removable server can be seen as follows. Let the ships be the customers and let the bridge be the server. The service time is the time it takes for a ship to pass under the bridge (we assume that there are physical constraints, so that only one ship can pass under the bridge at a time). The start-up time is the time it takes to open the bridge. Clearly, the cost structure here is the same as in the $M/G/1$ queueing system under consideration. Just let K be the expected (discounted) cost for halting the traffic on the bridge while a ship passes under it, and let R_1 and R_2 be such that they represent the direct cost for opening and closing the bridge plus the cost for halting the traffic on the bridge, while the bridge is being opened and closed, respectively. Notice that the holding cost function is linear.

Computer Time-Sharing Control:

Consider a company which has only one computer, but several terminals. The jobs originating from the terminals are the on-line jobs, and the jobs delivered to the operating room are the off-line jobs. The

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on-line jobs have priority over the off-line jobs. In fact, there is a cost incurred at a rate h for each on-line job which is kept waiting, while there are no costs associated with keeping off-line jobs waiting. However, a cost is incurred at a rate r while the computer does not process off-line jobs. If the computer is processing an off-line job when an on-line job arrives, the off-line job may be thrown out of the computer so that it can start processing the on-line job. When a job is thrown out of the computer, its entire memory content is transferred to an auxiliary memory device such that the processing of the job may be resumed later. If an off-line job is always thrown out of the computer when an on-line job arrives, there may be an excessive shifting of data from the computer to the auxiliary memory device and vice versa. It may therefore be desirable to wait until a number of on-line jobs have arrived before throwing an off-line job out of the computer. The problem is to determine when an off-line job should be thrown out of the computer (if at all).

That this problem can be viewed as an $M/G/1$ queueing system with removable server, can be seen as follows. Let the on-line jobs be the customers, and let the computer be the server. The service time is the time it takes to execute an on-line job, and the start-up time is the time it takes to shift the memory content of the computer to an auxiliary memory device. Clearly, the cost structure here is the same as in the $M/G/1$ queueing system under consideration. Just let K be the expected (discounted) cost for not using the computer for off-line jobs while an on-line job is being processed, and let R_1 and R_2 be such that they represent the cost for not using the computer for off-line jobs while

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data is shifted from the computer to an auxiliary device and vice versa, respectively. Notice that the holding cost function is linear.

Production Control:

Consider a manufacturing company which uses a high efficiency production line for the production of items of a certain type, say type A. The expected (discounted) cost for producing an item of type A is K_1. When an item of type A is completed, a reward K_2 is received. In order to produce an item of type A, an item of type B is needed. Items of type B arrive to the production line according to a stationary Poisson process. There is a cost for holding items of type B in the system, and it is incurred at a rate which is a non-decreasing, non-negative function of the number of items of type B present. This cost may represent the costs associated with storing and maintaining the items. When there are no items of type B present, there are two alternative actions available. The first one is simply to wait for items of type B to arrive. The second one is to switch the production at the production line to the production of items of type C. In order to do this, however, one has to set up the production line for the production of items of type C. Also, once the production line is set up for the production of items of type C, it has to be set up for the production of items of type A before the production of these items can be resumed. There is a setup cost associated with each setup. There is also a cost for not producing items of type C. This cost is incurred at a rate r while items of type C are not produced. The problem is to determine when the production at the production line should be switched from the production of one type of items to the production of

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another type of items (if at all).

That this problem can be viewed as an M/G/1 queueing system with removable server can be seen as follows. Let the items of type B be the customers, and let the production line be the server. The service time is the time it takes to produce an item of type A, and the start-up time is the time it takes to set up the production line for the production of items of type A. Clearly, the cost structure here is the same as in the M/G/1 queueing system under consideration. First let K represent the sum of the service cost, the product completion reward and the cost for not producing items of type C when an item of type A is produced. Also, let R_1 and R_2 represent the setup costs plus the cost for not producing items of type C while the production line is being set up for the production of items of type A and for the production of items of type C, respectively. Notice that the holding cost function may be non-linear.

1.3 Some Terminology.

We are interested in showing that certain simple intuitive types of policies are optimal. These policies are the hysteretic[*] policies. A policy is called hysteretic if there are two integers, say m and n $(n \leq m)$, such that the server is always turned on (or kept on) when the number of customers in the system is greater than or equal to m, and such that he is always turned off (or kept off) when the number of customers in the system is less than or equal to n. This policy is denoted by $\pi(n,m)$. The numbers m and n are the upper and lower

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[*] Confer with Gebhard (1966).

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intervention points, respectively.

If the lower intervention point is less than zero, or if the upper intervention point is equal to plus infinity, then the hysteretic policy is degenerate. Otherwise, the policy is non-degenerate. Hysteretic policies whose upper intervention points are finite and lower intervention points are less than one, are called natural hysteretic policies. The different types of hysteretic policies are pictured in Figure 1.

The aim of this study is to prove that there always exists a hysteretic policy which is optimal, and to give the conditions for when the various types of hysteretic policies are optimal. For the case where the holding cost function is linear, especially explicit and convenient results are obtained.

1.4 Previous Studies of the Problem.

As mentioned before, Yadin and Naor (1963) were the first ones to study the M/G/1 queueing system with removable server. They examined the steady-state behavior of the system, given that a natural non-degenerate hysteretic policy is used. Using a linear holding cost function, they found the value of the upper intervention point which minimizes the expected cost rate in steady-state.

Heyman (1968) was the first one to consider the optimal control of the M/G/1 queueing system with removable server. As with Yadin and Naor, he assumed a linear holding cost function. In addition, he assumed that the start-up times were zero. He considered the problem both with and without discounting, and proved the existence of a hysteretic optimal policy. However, his proofs were incomplete.

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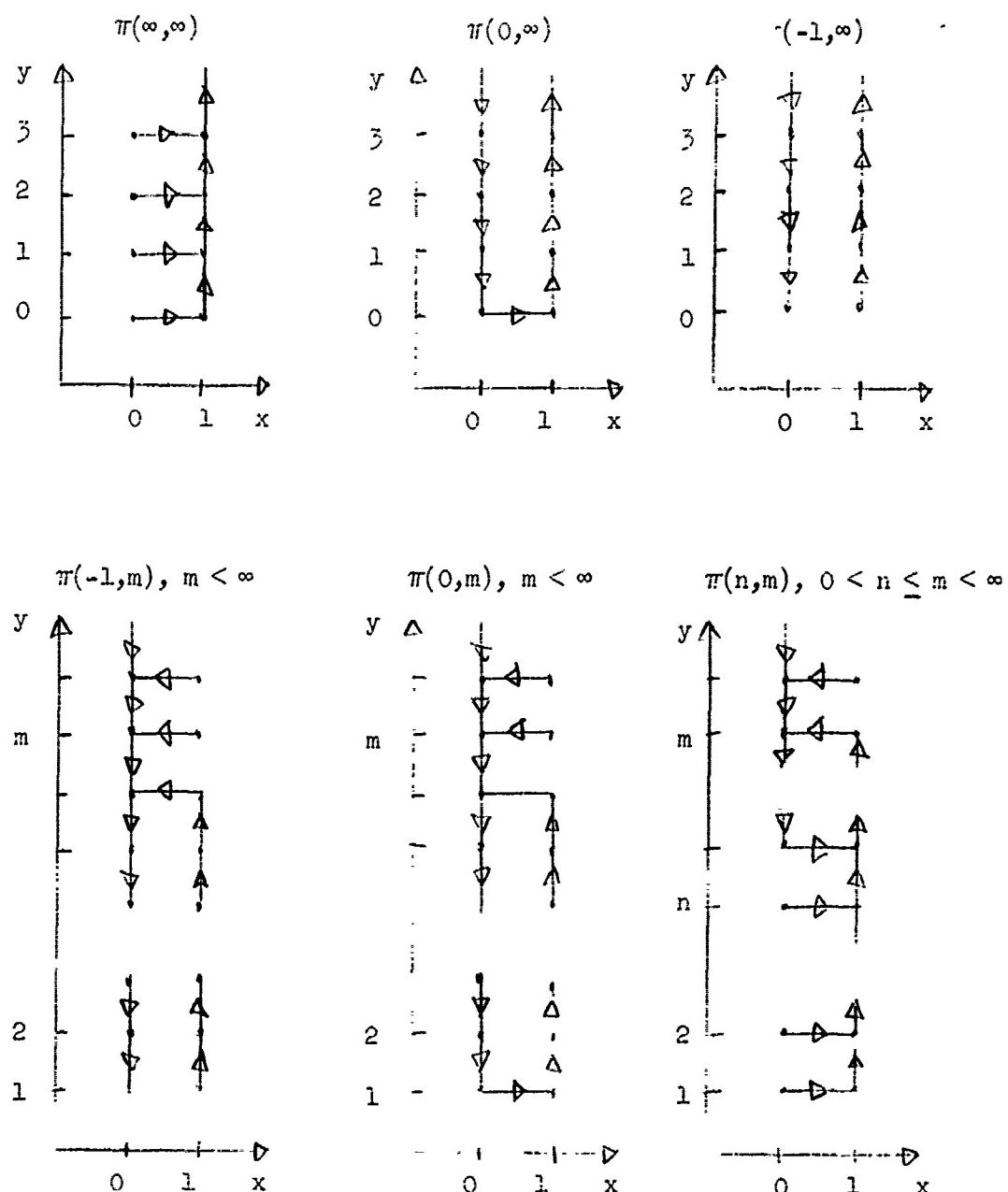
Figure 1. The types of hysteretic policies, where the x-axis indicates the status of the server, the y-axis indicates the number of customers in the system, and the arrows indicate how the system moves.

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Sobel (1969) considered the GI/G/1 queueing system with removable server. However, he used a cost structure which, it seems, would only be natural if the GI/G/1 system were in fact an M/G/1 system. He used the average cost criterion. Under some fairly weak conditions, he proved that there is a non-degenerate hysteretic policy which is optimal among all stationary policies.

Bell (1971) considered the same problem as Heyman, but only with discounting. He completed Heyman's proofs, and also gave an efficient algorithm for finding an optimal policy.

Blackburn (1971) independently obtained results similar to those of Bell. He also considered the more general case where the holding cost function is any non-negative, non-decreasing, convex function with a bounded slope. He used discounting, and under certain weak conditions proved that there is a non-degenerate hysteretic policy. However, the present author has found that his proof was incomplete at one point (namely in the proof of Lemma 18, Chapter 3). Intuitively, the result seems to be true, so it is still hoped that the proof can be completed.

Reed (1974a) also considered the M/G/1 queueing system with removable server. He used a new approach to the problem and derived similar, but somewhat more explicit results than those of Bell and Blackburn. Later, Reed (1974b) extended his previous results to cover the case of non-instantaneous start-up and shut-down times.

Recently, Deb (1976) considered the M/G/1 queueing system with removable server (actually he considered bulk service, but by letting the bulk size be equal to one, his problem becomes the same as ours). He allowed a general non-negative, non-decreasing holding cost function,

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but assumed instantaneous start-ups. His main result was that there exists a natural non-degenerate hysteretic policy which is optimal if the slope of the holding cost function is bounded below by a certain constant.

Other variants of the $M/G/1$ queueing system have been considered by Bell (1973), Blackburn (1972), Tijms (1975) and Levy and Yechiali (1975). In particular, Bell considered the system with several customer classes, Blackburn considered the system with balking and reneging, Tijms considered the system where the service time of a customer becomes known when he enters the system, and Levy and Yechiali considered the system where the server is removed for a random period of time with a given distribution function.

2. A Semi-Markov Decision Process Formulation.

The $M/G/1$ queueing system with removable server can be formulated as a semi-Markov decision process. In order to do this, a state space, an action space for each state, a law of motion and a cost function must be specified. We first identify the decision epochs, the state of the system and the set of permissible actions.

The decision epochs are the epochs when customers arrive and depart with the exception of those arrivals which occur while the server is giving service to another customer. At each decision epoch, the state of the system is defined as the pair of integers indicating the number of customers in the system and the status of the server (the second integer being 1 if the server is on and 0 if he is off). Thus, the state space becomes $N_0 \times \{0,1\}$, where N_0 is the set of non-negative integers.

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At each decision epoch there are always two available actions, action 0 and action 1. Action 0 is to turn the server off (or keep him off if already off), and action 1 is to turn him on (or keep him on if already on). Thus, each action space becomes $\{0,1\}$.

The law of motion and the cost function are in principle determined now. In order to avoid any ambiguities, a formal description of the law of motion and the cost function is included below.

The law of motion, q, is the mapping from $N_0 \times \{0,1\} \times \{0,1\} \times N_0 \times \{0,1\} \times R$ into R, given by

$$
q(i,j,k,i',j',t) = \begin{cases}
1 & , \text{ if } i=i', \ j=1, \ k=j'=0, \ t \geq 0, \\[2ex]
1 - e^{-\lambda t} & , \text{ if } \begin{cases} i=0, \ j=i'=k=1, \ t \geq 0, \\[1ex] i'=i+1, \ j=j'=k=0, \ t \geq 0, \end{cases} \\[3ex]
\int_0^t \frac{(\lambda u)^{i'-i+1}}{(i'-i+1)!} e^{-\lambda u} dF(u), & \text{ if } i' \geq i-1, \ j=j'=k=1, \ t \geq 0, \\[3ex]
\int_0^t \frac{(\lambda u)^{i'-i}}{(i'-i)!} e^{-\lambda u} dG(u) & , \text{ if } i' \geq i, \ j=0, \ j'=k=1, \ t \geq 0, \\[3ex]
0 & , \text{ otherwise,}
\end{cases}
$$

for $i \in N_0$, $j \in \{0,1\}$, $k \in \{0,1\}$, $i' \in N_0$, $j' \in \{0,1\}$ and $t \in R$.

The cost function c is a mapping from $N_0 \times \{0,1\} \times R$ into R, given by

$$c(i,j,k,t) = \begin{cases} R_2 & , \text{ if } k < j, \ t \geq 0, \\[2mm] rt & , \text{ if } i = 0, \ j = k = 1, \ t \geq 0 \\[2mm] h(i) \cdot t & , \text{ if } j = k = 0, \ t \geq 0, \\[2mm] R_1 \cdot F(t) + \sum_{n \in N_0} h(i+n) \int_0^t (1-F(u)) \frac{(\lambda u)^{i+n}}{(i+n)!} e^{-\lambda u} du, \\ \qquad , \text{ if } i > 0, \ j = k = 1, \ t \geq 0, \\[2mm] K + \sum_{n \in N_0} h(i+n) \int_0^t (1-F(u)) \frac{(\lambda u)^{i+n}}{(i+n)!} e^{-\lambda u} du, \\ \qquad , \text{ if } j = 0, \ k = 1, \ t \geq 0, \\[2mm] 0 & , \text{ otherwise}, \end{cases}$$

for $i \in N_0$, $j \in \{0,1\}$, $k \in \{0,1\}$ and $t \in R$.

The interpretation of q and c are as follows. Consider a decision epoch. Suppose that the state of the system at that decision epoch is (i,j) and that the action taken there is k. Pick a state (i',j') and a time t. Then $q(i,j,k,i',j',t)$ is just the joint probability that the next decision epoch occurs within a time t and that the state of the system at that decision epoch is (i',j'). Furthermore, $c(i,j,k,t)$ is just the expected cost accumulated within time t after the first decision epoch considered here. We now introduce some general notation to be used later.

Let $\mathcal{P}$ and $\mathcal{D}$ denote the class of all policies and the class of stationary, deterministic policies, respectively. For each $\pi \in \mathcal{D}$, let φ_π denote the long-run expected average cost (per unit of time), given that the policy π is used (the start-state is irrelevent in this

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case). For each $\pi \in \mathcal{D}$, $i \in N_0$ and $j \in \{0,1\}$, let $w_\pi(i,j)$ denote the long-run expected cost in excess of what is indicated by φ_π, given that the start-state is (i,j) and that the policy π is used. Finally, for each $\pi \in \mathcal{P}$, $i \in N_0$ and $j \in \{0,1\}$, let $v_\pi(i,j)$ denote the total expected discounted cost given that the start-state is (i,j) and that the policy π is used.

A policy π is <u>average optimal</u> if it minimizes φ_π ($\pi \in \mathcal{D}$), and it is <u>undiscounted optimal</u> if it minimizes $w_\pi(i,j)$ for each $(i,j) \in N_0 \times \{0,1\}$ among all average optimal policies (in $\mathcal{D}$). A policy is <u>discounted optimal</u> if it minimizes $v_\pi(i,j)$ for each $(i,j) \in N_0 \times \{0,1\}$ ($\pi \in \mathcal{P}$).

In order to be able to determine whether a given policy is optimal or not, we will need some optimality conditions. Fortunately, the problem without discounting can be solved quite directly, so we need only consider the problem with discounting here.

Optimality conditions for semi-Markov decision processes were given by Orkenyi (1976). The important concepts of improvable and unimprovable policies were introduced there.

A policy is <u>improvable</u> if there is a start-state such that the expected discounted cost, given that start-state (and the policy under consideration), can be reduced by changing the first action chosen by the policy. A policy is <u>unimprovable</u> if it is not improvable.

More formally, for each $\pi \in \mathcal{D}$, let $\mathcal{D}(\pi)$ denote the set of (deterministic) policies which uses the same decision rule as π after the first decision epoch. A policy π^* in $\mathcal{D}$ then is unimprovable if

$$v_{\pi^*}(i,j) \le v_\pi(i,j), \quad \text{for } i \in N_0, j \in \{0,1\}, \pi \in \mathcal{D}(\pi^*) .$$

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Orkenyi (1976) also showed that if a policy π^* in $\mathcal{D}$ is improvable, then there is a policy $\pi \in \mathcal{D}$ which is an improvement over π^*. More specifically, let π' be a policy in $\mathcal{D}(\pi^*)$ such that

$$v_{\pi'}(i,j) \leq v_{\pi^*}(i,j), \quad \text{for} \quad i \in N_0, \ j \in \{0,1\} \ ,$$

with a strict inequality for some state (i,j). Let π be the policy in $\mathcal{D}$ such that it uses the same decision rule as π' does at the first decision epoch. Then a theorem by Orkenyi (1976) says that

$$v_{\pi}(i,j) \leq v_{\pi'}(i,j), \quad \text{for} \quad i \in N_0, \ j \in \{0,1\} \ ,$$

and π is an improvement over π^*. This theorem is referred to as the policy improvement theorem.

Clearly, an optimal policy must be unimprovable. But an unimprovable policy need not always be optimal. Conditions ensuring that an unimprovable policy is optimal are given by Orkenyi (1976). Chapter 4 there contains a discussion of the optimality of unimprovable policies for the M/G/1 queueing system with removable server.

It is convenient to introduce the following general notation here. For any random variable T, $E_{\pi,s}\{T\}$ denotes the expected value of T given the policy π and start-state s.

3. The Case of Linear Holding Cost Function.

In this section we consider the case where the holding cost function is linear. This case has been studied extensively before. Reed (1974a), (1974b) has given a characterization of the optimal policies. Bell (1971) and Blackburn (1971) have given algorithms for finding an optimal policy. Here, some new and stronger results are presented. The emphasis is on

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obtaining results which are explicit and easy to use. The problem is
considered both with and without discounting.

3.1 The Undiscounted Case.

The problem is somewhat easier without discounting, so this case
is considered first. The optimality criterion is the undiscounted cri-
terion. We begin by obtaining some preliminary results.

3.1.1 Preliminaries.

Recall that λ is the arrival rate, μ is the service rate and
$\rho \ (= \lambda/\mu)$ is the load on the system. Let ζ, η and γ be defined
by

$$\zeta = \int_0^\infty t \, dG(t) \ ,$$

$$\eta = \int_0^\infty t^2 \, dG(t) \ ,$$

and

$$\gamma = \int_0^\infty t^2 \, dF(t) \ .$$

In words, ζ is the expected start-up time, η is the second moment
of the start-up time, and γ is the second moment of the service time.
We assume that these quantities are finite.

Let T denote the time until the state $(0,1)$ is reached, and
define κ and ν by

$$\kappa = E_{\pi(-1,0),(1,1)}(T) \ ,$$

and

$$\nu = E_{\pi(-1,0),(0,0)}(T) \ .$$

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By conditioning on the time until the second decision epoch and the state of the system at that epoch, we obtain

$$\kappa = \frac{1}{\mu} + \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t} \cdot i\kappa \cdot dF(t)$$

$$= \frac{1}{\mu} + \frac{\lambda}{\mu} \kappa \ ,$$

and

$$\nu = \zeta + \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t} \cdot i\kappa \cdot dG(t)$$

$$= \zeta + \lambda \zeta \kappa \ .$$

This implies that

$$\kappa = \frac{1}{\mu - \lambda} = \frac{1}{\mu} \cdot \frac{1}{1-\rho} \ ,$$

and

$$\nu = \frac{\zeta \mu}{\mu - \lambda} = \zeta \cdot \frac{1}{1-\rho} \ .$$

From this, we obtain

$$E_{\pi(-1,m),(0,0)}(T) = E_{\pi(0,m),(0,0)}(T)$$

$$= \frac{m}{\lambda} + \nu + m\kappa$$

$$= \frac{1}{1-\rho} \left(\frac{m}{\lambda} + \zeta \right), \quad \text{for} \quad m \in N_0 \ .$$

Let H denote the holding cost incurred until the state (0,1) is reached. Letting h denote the holding cost rate for each customer, then for each $i \in N_0$,

$$E_{\pi(-1,0),(i,0)}(H)$$

$$= \sum_{j=1}^{i} (E_{\pi(-1,0),(1,0)}(H) + (j-1)Kh)$$

$$= i \cdot E_{\pi(-1,0),(1,0)}(H) + \frac{1}{2} i(i-1)Kh \ .$$

By conditioning on the time until the second decision epoch and the state of the system at that epoch, we obtain

$$E_{\pi(-1,0),(1,0)}(H) = h \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t}(1-F(t)) \cdot (i+1)hdt$$

$$+ \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t}\{i \cdot E_{\pi(0,1),(1,0)}(H) + \frac{1}{2} i(i-1)Kh\}dF(t)$$

$$= \rho \cdot E_{\pi(0,1),(1,0)}(H) + (\frac{1}{\mu} + \frac{1}{2} \frac{\lambda \gamma}{1-\rho})h$$

and

$$E_{\pi(-1,0),(0,0)}(H) = h \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t}(1-G(t)) \cdot i \cdot hdt$$

$$+ \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t}\{i \cdot E_{\pi(0,1),(1,0)}(H) + \frac{1}{2} i(i-1)Kh\}dG(t)$$

$$= \lambda \zeta \cdot E_{\pi(0,1),(1,0)}(H) + \frac{1}{2} \cdot \frac{\lambda \eta}{1-\rho} \cdot h \ .$$

This implies that

$$E_{\pi(-1,0),(1,0)}(H) = (\frac{1}{\mu} \cdot \frac{1}{1-\rho} + \frac{1}{2} \frac{\lambda \gamma}{(1-\rho)^2})h \ ,$$

and

$$E_{\pi(-1,0),(0,0)}(H) = (\frac{\rho}{1-\rho} \zeta + \frac{1}{2} \frac{\lambda^2 \gamma}{(1-\rho)^2} \zeta + \frac{1}{2} \frac{\lambda \eta}{(1-\rho)^2})h \ .$$

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From this, we obtain

$$E_{\pi(-1,m),(0,0)}\{H\} = E_{\pi(0,m),(0,0)}\{H\}$$

$$= \frac{h}{\lambda} \sum_{i=1}^{m-1} i$$

$$+ E_{\pi(-1,0),(0,0)}\{H\} + m \cdot h \cdot \nu$$

$$+ E_{\pi(-1,0),(m\rho)}\{H\}$$

$$= \frac{1}{2} m(m-1)\frac{h}{\lambda}$$

$$+ (\frac{\rho}{1-\rho} \cdot \zeta + \frac{1}{2} \frac{\lambda^2 \gamma}{(1-\rho)^2} \cdot \zeta + \frac{1}{2} \frac{\lambda \eta}{(1-\rho)^2})h + \frac{\zeta}{1-\rho} \cdot m \cdot h$$

$$+ \frac{1}{2} m(m-1) \cdot \frac{1}{\mu} \cdot \frac{1}{1-\rho} \cdot h + m(\frac{1}{\mu} \cdot \frac{1}{1-\rho} + \frac{1}{2} \frac{\lambda \gamma}{(1-\rho)^2})h$$

$$= \frac{1}{2} m(m-1) \cdot \frac{1}{1-\rho} \cdot \frac{h}{\lambda} + m(\zeta + \frac{1}{\mu} + \frac{1}{2} \frac{\lambda \gamma}{1-\rho})\frac{h}{1-\rho}$$

$$+ (\rho \cdot \zeta + \frac{1}{2} \frac{\lambda^2 \gamma}{1-\rho} \zeta + \frac{1}{2} \frac{\lambda \eta}{1-\rho})\frac{h}{1-\rho}, \quad \text{for} \quad m \in N_0 .$$

Let C denote the cost incurred until the state (0,1) is reached. Then, for each $m \in N_0$,

$$\varphi_{\pi(-1,m)} = \lambda(1-\rho) \cdot \{\frac{r}{\lambda} + (\frac{1}{\mu} \cdot \frac{1}{1-\rho} + \frac{1}{2} \frac{\lambda \gamma}{(1-\rho)^2})h\}$$

$$= r(1-\rho) + (\rho + \frac{1}{2} \frac{\lambda^2 \gamma}{1-\rho})h ,$$

and

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$$\varphi_{\pi(0,m)} = \frac{R_1 + R_2 + E_{\pi(0,m),(0,0)}{\{H\}}}{E_{\pi(0,m),(0,0)}{\{T\}}}$$

$$= \frac{\lambda(1-\rho)}{m+\lambda\zeta} \left(R_1 + R_2 + E_{\pi(0,m),(0,0)}{\{H\}}\right)$$

$$= \frac{1}{m+\lambda\zeta} \cdot \{(R_1 + R_2)\lambda(1-\rho) + (\tfrac{1}{2} m(m-1) + m(\lambda\zeta + \rho + \tfrac{1}{2}\frac{\lambda^2\gamma}{1-\rho})$$

$$\div \lambda\rho\zeta + \tfrac{1}{2}\frac{\lambda^3\gamma}{1-\rho}\zeta + \tfrac{1}{2}\frac{\lambda^2\eta}{1-\rho})h\} \ .$$

3.1.2 The Optimal Value of m in $\pi(-1,m)$.

We will now consider the optimal value of m in $\pi(-1,m)$. Let m denote this value. That m exists is shown in Section 4.1. It is given by

$$w_{\pi(-1,m')}(0,0) \leq w_{\pi(-1,m)}(0,0), \quad \text{for}\ \ m \in N_0 \ .$$

Now

$$w_{\pi(-1,m)}(0,0) = E_{\pi(-1,m),(0,0)}{\{C\}} - \varphi_{\pi(-1,0)} \cdot E_{\pi(-1,m),(0,0)}{\{T\}}$$

$$+ w_{\pi(-1,0)}(0,1), \quad \text{for}\ \ m \in N_0 \ ,$$

so

$$w_{\pi(-1,m+1)}(0,0) - w_{\pi(-1,m)}(0,0)$$

$$= E_{\pi(-1,m+1),(0,0)}{\{C\}} - E_{\pi(-1,m),(0,0)}{\{C\}}$$

$$- \varphi_{\pi(-1,0)} \cdot (E_{\pi(-1,m+1),(0,0)}{\{T\}} - E_{\pi(-1,m),(0,0)}{\{T\}})$$

$$= \frac{m}{1-\rho} \frac{h}{\lambda} + (\zeta + \frac{1}{\mu} + \frac{1}{2}\frac{\lambda\gamma}{1-\rho})\frac{h}{1-\rho}$$

$$- (r(1-\rho) + (\rho + \frac{1}{2} \cdot \frac{\lambda^2 \gamma}{1-\rho})h) \cdot \frac{1}{\lambda(1-\rho)}$$

$$= \frac{1}{1-\rho} \frac{h}{\lambda}(m + \lambda\zeta - \frac{r}{h}(1-\rho)), \quad \text{for} \quad m \in N_0 .$$

Thus

$$m' = \min\{m \in N_0 | m \geq \frac{r}{h}(1-\rho) - \lambda\zeta\} .$$

3.1.3 The Optimal Value of m in $\pi(0,m)$.

We will now find the optimal value of m in $\pi(0,m)$. Let m''
denote this value. Then m'' is given by

$$\varphi_{\pi(0,m'')} \leq \varphi_{\pi(0,m)}, \quad \text{for} \quad m \in N_0 .$$

Now

$$\varphi_{\pi(0,m+1)} - \varphi_{\pi(0,m)} = \frac{\lambda(1-\rho)}{(m+1+\lambda\zeta)(m+\lambda\zeta)} \cdot \{(m+\lambda\zeta) \cdot E_{\pi(0,m+1),(0,0)}\{H\}$$

$$- (m+1+\lambda\zeta) \cdot E_{\pi(0,m),(0,0)}\{H\} - (R_1 + R_2)\}$$

$$= \frac{h}{(m+1+\lambda\zeta)(m+\lambda\zeta)} \cdot \{(m+\lambda\zeta)(\frac{1}{2} m(m+1)+(m+1)(\lambda\zeta+\rho+\frac{1}{2}\frac{\lambda^2\gamma}{1-\rho}))$$

$$- (m+1+\lambda\zeta)(\frac{1}{2} m(m-1) + m(\lambda\zeta + \rho + \frac{1}{2}\frac{\lambda^2\gamma}{1-\rho}))$$

$$- (\frac{\lambda}{h}(1-\rho)(R_1 + R_2) + \lambda\rho\zeta + \frac{1}{2}\frac{\lambda^3\gamma}{1-\rho}\zeta + \frac{1}{2}\frac{\lambda^2\eta}{1-\rho})\}$$

$$= \frac{h}{(m+1+\lambda\zeta)(m+\lambda\zeta)} \cdot \{\frac{1}{2} m^2 + \frac{1}{2}(1+2\lambda\zeta)m$$

$$+ \lambda\zeta(\lambda\zeta + \rho + \frac{1}{2}\frac{\lambda^2\gamma}{1-\rho} - \rho - \frac{1}{2}\frac{\lambda^2\gamma}{1-\rho} - \frac{1}{2}\frac{\lambda}{1-\rho}\frac{\eta}{\zeta})$$

$$- \frac{\lambda}{h}(1-\rho)(R_1 + R_2)\}$$

$$= \frac{h}{(m+1+\lambda\zeta)(m+\lambda\zeta)} \cdot \{\frac{1}{2}(m + \lambda\zeta + \frac{1}{2})^2 - \frac{1}{2}(\lambda\zeta + \frac{1}{2})^2$$

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$$+ \lambda\zeta(\lambda\zeta - \frac{1}{2}\frac{\lambda}{1-\rho}\frac{n}{\zeta}) - \frac{2\lambda}{h}(1-\rho)(R_1 + R_2)\}$$

$$= \frac{h}{2(m+1+\lambda\zeta)(m+\lambda\zeta)} \cdot \{(m+\lambda\zeta+\frac{1}{2})^2 + (\lambda\zeta)^2 - \lambda\zeta$$

$$- \frac{1}{4} - \frac{\lambda^2\eta}{1-\rho} - \frac{2\lambda}{h}(1-\rho)(R_1 + R_2)\}$$

$$= \frac{h}{2(m+1+\lambda\zeta)(m+\lambda\zeta)} \cdot \{(m + \lambda\zeta + \frac{1}{2})^2 + (\lambda\zeta - \frac{1}{2})^2 - \frac{\lambda^2\eta}{1-\rho}$$

$$- \frac{2\lambda}{h}(1-\rho)(R_1 + R_2) - \frac{1}{2}\}, \quad \text{for} \quad m \in N_0 \ .$$

Thus, we obtain

$$m'' = \min\{m \in N_0 \, | \, m \geq -\frac{1}{2} - \lambda\zeta + \sqrt{\frac{2\lambda(1-\rho)}{h} \cdot (R_1+R_2) + \frac{\lambda^2\eta}{1-\rho} - (\lambda\zeta - \frac{1}{2})^2 + \frac{1}{2}}\} \ .$$

3.1.4 Characterization of the Optimal Policies.

It is proven in Section 4.1 for the general case of non-decreasing, convex holding cost function that either a policy $\pi(-1,m)(m < \infty)$ or a policy $\pi(0,m)(m < \infty)$ is undiscounted optimal, depending on which is average optimal. In principle therefore, all one has to do to find the (an) optimal policy is to compute m'' (by using the formulae in Section 3.1.3), compute the long run expected average cost, given that the policy $\pi(0,m'')$ is used (by using the formulae in Section 3.1.1), and compare it with the long run expected average cost, given that the policy $\pi(-1,m')$ is used.

3.2 The Discounted Case.

Here, we use the discounted cost criterion. The analysis becomes somewhat different from that in the preceding section. One reason is that it is possible to reformulate the problem so that the holding costs do not need to be considered explicitly.

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3.2.1 Elimination of the Holding Costs from the Analysis.

Bell (1971) suggested a reformulation of the original problem such that the holding costs would become bounded. Here, we show that the original problem can be reformulated in such a way that the holding costs are eliminated from the analysis altogether.

Since the holding cost function is linear, the total expected discounted holding cost is equal to the sum of the expected discounted holding cost for the respective customers. Let as before h denote the individual holding cost rate. For each $m \in N$, let t_{2n} and t_{2n+1} denote the times when the nth customer arrives and departs, respectively. Then the total expected discounted holding cost is

$$E\{\sum_{n \in N} h \int_{t_{2n}}^{t_{2n+1}} e^{-\alpha t} dt\}$$

$$= E\{\sum_{n \in N} \frac{h}{\alpha}(e^{-\alpha t_{2n}} - e^{-\alpha t_{2n+1}})\}$$

$$= E\{\sum_{n \in N} \frac{h}{\alpha} e^{-\alpha t_{2n}}\} - E\{\sum_{n \in N} \frac{h}{\alpha} e^{-\alpha t_{2n}}\} .$$

Since the arrival process is not affected by the policy in use, the first term in the above expression is neither. Therefore, it may be neglected when searching for an optimal policy. The second term does depend on the policy in use, and therefore cannot be neglected.

Suppose now that at each service completion a reward h/α would be received. Clearly, the expected discounted cost arising from the service completion rewards would just be equal to the second term in the above expression. Therefore, the original problem must be equivalent

to the problem in which a reward h/α is received at each service completion instead of incurring a holding cost at a rate h for each customer in the system. Thus, since the service completion reward may be included in the service cost, we may assume <u>without loss of generality</u> that there are no holding costs. This will now be done.

3.2.2 <u>Preliminaries</u>.

Let σ, ω and ξ be given by

$$\sigma = \frac{\lambda}{\lambda + \alpha} \ ,$$

$$\omega = \int_0^\infty e^{-\alpha t} dF(t) \ ,$$

$$\xi = \int_0^\infty e^{-\alpha t} dG(t) \ .$$

In words, σ is the Laplace transform of the inter-arrival times, ω is the Laplace transform of the service times and ξ is the Laplace transform of the start-up times.

Let as before T denote the time until the state $(0,1)$ is reached, and define Ψ and X by

$$\Psi = E_{\pi(-1,0),(1,1)}\{e^{-\alpha T}\} \ ,$$

and

$$X = E_{\pi(-1,0),(0,0)}\{e^{-\alpha T}\} \ .$$

By conditioning on the time until the second decision epoch and the state of the system at that epoch, we obtain

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$$\psi = \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-(\alpha+\lambda)t} \cdot \psi^i \cdot dF(t)$$

$$= \int_0^\infty e^{-(\alpha+\lambda-\lambda\psi)} dF(t) ,$$

and

$$X = \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-(\alpha+\lambda)t} \cdot \psi^i \cdot dG(t)$$

$$= \int_0^\infty e^{-(\alpha+\lambda-\lambda\psi)} dG(t) .$$

Since $\rho < 1$, ψ is the unique solution of the above equation in the interval $[0,1]$. This can be seen as follows.

Let g be the function from $[0,1]$ into R, given by

$$g(x) = x - \int_0^\infty e^{-(\alpha+\lambda-\lambda x)t} dF(t), \quad \text{for} \quad x \in [0,1] .$$

Taking the derivative, we obtain

$$g'(x) = 1 - \lambda \int_0^\infty t e^{-(\alpha+\lambda-\lambda x)t} dF(t)$$

$$\geq 1 - \lambda \int_0^\infty t \, dF(t)$$

$$> 0, \quad \text{for} \quad x \in (0,1) .$$

Also,

$$g(0) = -\int_0^\infty e^{-(\alpha+\lambda)t} dF(t) < 0 ,$$

and

$$g(1) = 1 - \int_0^\infty e^{-\alpha t} dF(t) \geq 0 .$$

Therefore, by the mean value theorem, the equation

$$g(x) = 0$$

has a unique solution in the closed interval $[0,1]$.

We will now consider the costs. It is useful to introduce the following quantities. Let

$$A = R_2 - \frac{K}{1-\omega} \, ,$$

$$B = - R_1 - \frac{K}{1-\omega} \, ,$$

and

$$C = \frac{\alpha R_2 - r}{\lambda(1-\psi)} - \frac{K}{1-\omega} \, ,$$

$$D = \frac{r}{\alpha} - \frac{K}{1-\omega} \, .$$

Also, let Z denote the total discounted cost incurred until the state $(0,1)$ is reached. Then, for each $i \in N_0$,

$$E_{\pi(-1,0),(i,1)}\{Z\} = \frac{1-\psi^i}{1-\omega} \cdot K \, .$$

By conditioning on the time until the second decision epoch and the state of the system at that epoch, we obtain

$$E_{\pi(-1,m),(m,0)}\{Z\} = \sum_{i \in N_0} \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t} \cdot e^{-\alpha t} (\frac{1-\psi^{i+m}}{1-\omega} K) dG(t)$$

$$= (\xi - \chi\psi^m) \cdot \frac{K}{1-\omega}, \quad \text{for} \quad m \in N_0 \, .$$

Using this, we obtain

$$v_{\pi(-1,m)}(0,0) = \sigma^m\left(\frac{K}{1-\omega}(\xi - X\psi^m) + \xi R_1\right)$$

$$+ (\sigma\psi)^m \cdot X\left(\frac{r}{\lambda+\alpha} + \sigma\frac{1-\psi}{1-\omega}K\right)(1-\sigma\psi)^{-1}$$

$$= \sigma^m\xi\left(\frac{K}{1-\omega} + R_1\right) + (\sigma\psi)^m \cdot X\left(\frac{r}{\lambda+\alpha} - \frac{1-\sigma}{1-\omega}K\right)(1-\sigma\psi)^{-1}$$

$$= -B\xi\sigma^m + D\frac{1-\sigma}{1-\sigma\psi} \cdot X(\sigma\psi)^m, \quad \text{for} \quad m \in N_0 ,$$

and

$$v_{\pi(0,m)}(0,0) = \sigma^m\left(\frac{K}{1-\omega}(\xi - X\psi^m) + \xi R_1 + X\psi^m R_2\right)(1-X(\sigma\psi)^m)^{-1}$$

$$= \sigma^m(-\xi B + AX\psi^m)(1-X(\sigma\psi)^m)^{-1}$$

$$= -A + \frac{A-\xi BC^m}{1-X(\sigma\psi)^m}, \quad \text{for} \quad m \in N_0 .$$

3.2.3 The Optimal Value of m in $\pi(-1,m)$.

From the preceding section, we obtain

$$v_{\pi(-1,m+1)}(0,0) - v_{\pi(-1,m)}(0,0)$$

$$= \xi B(1-\sigma)\sigma^m - D\frac{1-\sigma}{1-\sigma\psi}X(1-\sigma\psi)(\sigma\psi)^m$$

$$= (1-\sigma)\sigma^m(\xi B - DX\psi^m) .$$

Suppose first that $D \leq 0$. Then the sign of

$$v_{\pi(-1,m+1)}(0,0) - v_{\pi(-1,m)}(0,0)$$

cannot change from being negative to being positive, so the optimal
value of m in $\pi(-1,m)$ must be either 0 or ∞. Now

$$v_{\pi(-1,0)} = -B\xi + \frac{1-\sigma}{1-\sigma\psi}XD ,$$

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and

$$v_{\pi(-1,\infty)} = 0 \ .$$

Therefore, the optimal value of m in $\pi(-1,m)$ is determined by the sign of

$$\frac{1-\sigma}{1-\sigma\psi} \, XD - \xi B \ .$$

Suppose now that $D > 0$. If $B \leq 0$, then

$$v_{\pi(-1,m+1)}(0,0) - v_{\pi(-1,m)}(0,0)$$

is negative for all $m \in N_0$, so the optimal value of m in $\pi(-1,m)$ is infinity. If $B > 0$, then

$$v_{\pi(-1,m+1)}(0,0) - v_{\pi(-1,m)}(0,0)$$

changes sign from negative to positive exactly once. Therefore, in this case, the optimal value of m in $\pi(-1,m)$ is

$$m = \frac{\log(\frac{\xi B}{XD})}{\log \psi} \ ,$$

rounded up to the nearest non-negative integer.

3.2.4 <u>The Optimal Value of m in $\pi(0,m)$.</u>

From Section 3.2.2, we obtain

$$v_{\pi(0,m+1)}(0,0) - v_{\pi(0,m)}(0,0)$$

$$= \frac{A - \xi B \sigma^{m+1}}{1 - X(\sigma\psi)^{m+1}} - \frac{A - \xi B \sigma^m}{1 - X(\sigma\psi)^m}$$

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$$= \frac{(\sigma\psi)^m (1-\sigma\psi)\xi B}{(1-X(\sigma\psi)^m)(1-X(\sigma\psi)^{m+1})} \cdot \left(\frac{1-\sigma}{1-\sigma\psi}\psi^{-m} + X\frac{1-\psi}{1-\sigma\psi}\sigma\sigma^m - \frac{XA}{\xi B}\right),$$

for $m \in N_0$.

Let f be the mapping from N_0 into R_1 given by

$$f(m) = \frac{1-\sigma}{1-\sigma\psi}\psi^{-m} + X\sigma\frac{1-\psi}{1-\sigma\psi}\sigma^m, \quad \text{for} \quad m \in N_0 .$$

Then

$$v_{\pi(0,m+1)}(0,0) - v_{\pi(0,m)}(0,0)$$

$$= \frac{(1-\sigma\psi)(\sigma\psi)^m(\xi Bf(m)-XA)}{(1-X(\sigma\psi)^m)(1-X(\sigma\psi)^{m+1})}, \quad \text{for} \quad m \in N_0 .$$

Notice that f is an increasing function, since

$$f(m) - f(m-1) = \frac{1-\sigma}{1-\sigma\psi}(1-\psi)\psi^{-m} - X\sigma\frac{1-\psi}{1-\sigma\psi}(1-\sigma)\sigma^{m-1}$$

$$= \frac{(1-\sigma)(1-\psi)}{1-\sigma\psi}(\psi^{-m} - X\sigma^m), \quad \text{for} \quad m \in N .$$

Suppose first that $B \leq 0$. Then the sign of

$$v_{\pi(0,m+1)}(0,0) - v_{\pi(0,m)}(0,0)$$

cannot change from negative to positive, as m increases, so the optimal value of m in $\pi(0,m)$ is either 0 or ∞. Now

$$v_{\pi(0,0)}(0,0) = \frac{XA-\xi B}{1-X} ,$$

and

$$v_{\pi(0,\infty)}(0,0) = 0 .$$

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Therefore, the optimal value of m in $\pi(0,m)$ is determined by the sign of

$$\lambda A - \xi B \ .$$

Suppose now that $B > 0$. This implies that $A > 0$, and

$$v_{\pi(0,m+1)}(0,0) - v_{\pi(0,m)}(0,0)$$

changes sign from negative to positive exactly once as m is increased. Therefore, the optimal value of m in $\pi(0,m)$ is

$$m = \min\{m \in N_0 \mid f(m) \geq \frac{\lambda A}{\xi B}\} \ .$$

3.2.5 Characterization of the Optimal Policies.

We now will show that a hysteretic policy is optimal, and specify when the different types of hysteretic policies are optimal. Since we have a semi-Markov decision process with bounded costs, an unimprovable policy is always optimal. Therefore, we will prove that a policy is optimal by proving that it is unimprovable.

Lemma 1: If $A \leq 0$, then $\pi(\infty,\infty)$ is optimal.

Proof: We have to show that

$$v_{\pi(\infty,\infty)}(i,j) \leq v_{\pi}(i,j), \quad \text{for} \quad i \in N_0, \ j \in \{0,1\}, \ \pi \in \mathcal{D}(\pi(\infty,\infty)) \ .$$

Consider the states in which the server is off. If a policy $\pi \in \mathcal{D}(\pi(\infty,\infty))$ starts with turning the server on when the start-state is $(i,0)$, then

$$v_{\pi}(i,0) = \xi(R_1 + R_2) \ .$$

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Since $R_1 + R_2 \geq 0$, we conclude that

$$v_{\pi(\infty,\infty)}(i,0) \leq v_\pi(i,0), \quad \text{for } i \in N_0, \; \pi \in \mathcal{D}(\pi(\infty,\infty)) .$$

Consider the states in which the server is on. If a policy $\pi \in \mathcal{D}(\pi(\infty,\infty))$ starts with keeping the server on when the start-state is $(i,1)$, then

$$v_\pi(0,1) = \frac{r}{\lambda+\alpha} + \sigma \cdot R_2 ,$$

and

$$v_\pi(i,1) = K + \omega R_2, \quad \text{for } i \in N .$$

Since $r \geq 0$ and $A \leq 0$, we conclude that

$$v_{\pi(\infty,\infty)}(i,1) \leq v_\pi(i,1), \quad \text{for } i \in N, \; \pi \in \mathcal{D}(\pi(\infty,\infty)) .$$

Thus, $\pi(\infty,\infty)$ is unimprovable and optimal.

$$Q.E.D.$$

Lemma 2: If $A \geq 0$, $B \leq 0$ and $C \leq 0$, then $\pi(0,\infty)$ is optimal.

Proof: We only have to show that

$$v_{\pi(0,\infty)}(i,j) \leq v_\pi(i,j), \quad \text{for } i \in N_0, \; j \in \{0,1\}, \; \pi \in \mathcal{D}(\pi(0,\infty)) .$$

Consider the states in which the server is off. Let $\pi \in \mathcal{D}(\pi(0,\infty))$ be the same policy as $\pi(0,\infty)$ except that it turns the server on when the start-state is $(i,0)$. Suppose that

$$v_\pi(i,0) < v_{\pi(0,\infty)}(i,0) .$$

By the policy improvement theorem,

$$v_{\pi(0,i)}(i,0) \leq v_\pi(i,0) .$$

This implies that

$$v_{\pi(0,i)}(i,0) < v_{\pi(0,\infty)}(i,0) \ .$$

But from Section 3.2.4,

$$v_{\pi(0,i)}(i,0) \geq v_{\pi(0,\infty)}(i,0) \ ,$$

since $B \leq 0$. This is a contradiction. Therefore,

$$v_{\pi(0,\infty)}(i,0) \leq v_{\pi}(i,0), \quad \text{for} \quad i \in N_0, \ \pi \in \mathcal{D}(\pi(0,\infty)) \ .$$

Consider the states in which the server is on and the system is not empty. If a policy $\pi \in \mathcal{D}(\pi(0,\infty))$ starts with turning the server off when the start-state is $(i,1)$, then

$$v_{\pi}(i,1) = R_2 \ .$$

Now

$$v_{\pi(0,\infty)}(i,1) = \frac{1-\psi^i}{1-\omega} K + \psi^i \cdot R_2 \ .$$

Clearly $A \geq 0$ implies that

$$v_{\pi}(i,1) \geq v_{\pi(0,\infty)}(i,1) \ .$$

Consider the state $(0,1)$. If a policy $\pi \in \mathcal{D}(\pi(0,\infty))$ starts with keeping the server on, then

$$v_{\pi}(0,1) = \frac{r}{\lambda+\alpha} + \sigma \cdot v_{\pi(0,\infty)}(1,1)$$

$$= \frac{r}{\lambda+\alpha} + \sigma(\frac{1-\psi}{1-\omega} K + \psi R_2) \ .$$

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Therefore,

$$v_\pi(0,1) - v_{\pi(0,\infty)}(0,1) = \frac{r}{\lambda+\alpha} + \sigma(1-\psi)(\frac{K}{1-\omega} - R_2) - (1-\sigma)R_2$$

$$= -\sigma(1-\psi)C$$

$$\geq 0 \ .$$

Thus, $\pi(0,\infty)$ is unimprovable and optimal.

<div align="right">Q.E.D.</div>

Lemma 3: If $A \geq 0$, $B \leq 0$ and $C \geq 0$, then $\pi(-1,\infty)$ is optimal.

Proof: We only have to show that

$$v_{\pi(0,\infty)}(i,j) \leq v_\pi(i,j), \quad \text{for} \quad i \in N_0, \ j \in \{0,1\}, \ \pi \in \mathcal{D}(\pi(-1,\infty)) \ .$$

Consider the states in which the server is off. Let $\pi \in \mathcal{D}(\pi(-1,\infty))$ be the same policy as $\pi(-1,\infty)$ except that it turns the server on when the start-state is $(i,0)$. Suppose that

$$v_\pi(i,0) < v_{\pi(-1,\infty)}(i,0) \ .$$

But

$$v_{\pi(-1,i)}(i,0) = v_\pi(i,0) \ ,$$

so

$$v_{\pi(-1,i)}(i,0) < v_{\pi(-1,\infty)}(i,0) \ .$$

From Section 3.2.3,

$$v_{\pi(-1,i)}(i,0) \geq v_{\pi(-1,\infty)}(i,0) \ ,$$

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since $B \leq 0$. This is a contradiction. Therefore,

$$v_{\pi(-1,\infty)}(i,0) \leq v_{\pi}(i,0), \quad \text{for} \quad i \in N_0, \ \pi \in \mathcal{D}(\pi(-1,\infty)) \ .$$

Consider the states in which the server is on. If a policy $\pi \in \mathcal{D}(\pi(-1,\infty))$ starts with turning the server off when the start-state is $(i,1)$, then

$$v_{\pi}(i,1) = R_2 \ .$$

Now

$$
\begin{aligned}
v_{\pi(-1,\infty)}(i,1) &= \frac{1-\psi^i}{1-\omega} K + \psi^i \Big(\frac{r}{\lambda+\alpha} + \sigma \frac{1-\psi}{1-\omega} K\Big)(1-\sigma\omega)^{-1} \\
&= \frac{K}{1-\omega} + \psi^i \Big(\frac{r}{\lambda+\alpha} - \frac{1-\sigma}{1-\omega} K\Big)(1-\sigma\omega)^{-1} \\
&= \frac{K}{1-\omega} + \frac{1-\sigma}{1-\sigma\psi} \cdot D \cdot \psi^i \\
&\leq \frac{K}{1-\omega} + \frac{1-\sigma}{1-\sigma\omega} \cdot D^+ \ .
\end{aligned}
$$

Therefore

$$v_{\pi}(i,1) - v_{\pi(-1,\infty)}(i,1) \geq A - \frac{1-\sigma}{1-\sigma\psi} \cdot D^+ \ .$$

If $D \leq 0$, then

$$v_{\pi}(i,1) \geq v_{\pi(-1,\infty)}(i,1) \ ,$$

since $A \geq 0$. If $D > 0$, then

$$
\begin{aligned}
v_{\pi}(i,1) - v_{\pi(-1,\infty)}(i,1) &\geq A - \frac{1-\sigma}{1-\sigma\psi} \cdot D \\
&= \sigma \cdot \frac{1-\psi}{1-\sigma\psi} \cdot C \\
&\geq 0 \ ,
\end{aligned}
$$

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so

$$v_{\pi}(i,1) \geq v_{\pi(-1,\infty)}(i,1) .$$

Thus, $\pi(-1,\infty)$ is unimprovable and optimal.

<div align="right">Q.E.D.</div>

Theorem 4: If $B > 0$, then a natural hysteretic policy is optimal.

Proof: Let m' be the (an) optimal value of m in $\pi(0,m)$. From Section 3.2, we know that m' is finite. Suppose first that $\pi(0,m')$ is at least as good as $\pi(-1,m)$ for all $m \in N_0$. Then $\pi(0,m')$ is optimal. To show this, we only need to show that

$$v_{\pi(0,m')}(i,j) \leq v_{\pi}(i,j), \quad \text{for} \quad i \in N_0, j \in \{0,1\}, \pi \in \mathcal{D}(\pi(0,m')) .$$

Consider the state $(0,1)$. Let $\pi \in \mathcal{D}(\pi(0,m'))$ be the same policy as $\pi(0,m')$ except that it keeps the server on in state $(0,1)$. Suppose that

$$v_{\pi}(0,1) < v_{\pi(0,m')}(0,1) .$$

By the policy improvement theorem,

$$v_{\pi(-1,m')}(0,1) \leq v_{\pi}(0,1) ,$$

so

$$v_{\pi(-1,m')}(0,1) < v_{\pi(-0,m')}(0,1) .$$

But we just assumed that this is not the case, so we conclude that

$$v_{\pi(0,m')}(0,1) \leq v_{\pi}(0,1), \quad \text{for} \quad \pi \in \mathcal{D}(\pi(0,m')) .$$

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Consider the states in which the server is off and the number of customers is less than m'. Let $\pi \in \mathcal{D}(\pi(0,m'))$ be the same policy as $\pi(0,m')$ except that it turns the server on in a state $(i,0)$, $i < m$. Suppose that

$$v_\pi(i,0) < v_{\pi(0,m')}(i,0) .$$

By the policy improvement theorem,

$$v_{\pi(0,i)}(i,0) \leq v_\pi(i,0) ,$$

so

$$v_{\pi(0,i)}(i,0) < v_{\pi(0,m')}(i,0) .$$

This is a contradiction, so we conclude that

$$v_{\pi(0,m')}(i,0) \leq v_\pi(i,0), \quad \text{for} \quad i < m', \ \pi \in \mathcal{D}(\pi(0,m')) .$$

Consider the states in which the server is off and there are at least m' customers present. Let $\pi \in \mathcal{D}(\pi(0,m'))$ be the same policy as $\pi(0,m')$ except that it keeps the server off in a state $(i,0)$, $i \geq m'$. From Section 3.2.4, we know that

$$v_{\pi(i+1)}(i,0) \geq v_{\pi(i)}(i,0) .$$

Now

$$v_{\pi(i)}(i,1) \geq v_{\pi(0,m')}(i,1) ,$$

so

$$v_\pi(i,0) \geq v_{\pi(0,m')}(i,0) .$$

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Thus

$$v_{\pi(0,m')}(i,0) \leq v_{\pi}(i,0), \quad \text{for} \quad i \geq m, \ \pi \in \mathcal{D}(\pi(0,m')) \ .$$

Consider the states in which the server is on and there is at least one customer present. Let $\pi \in \mathcal{D}(\pi(0,m'))$ be the same policy as $\pi(0,m')$ except that it turns off the server in a state $(i,1)$, $0 < i \leq m'$. Suppose that

$$v_{\pi}(i,1) < v_{\pi(0,m')}(i,1) \ .$$

By the policy improvement theorem,

$$v_{\pi(i,m')}(i,1) \leq v_{\pi}(i,1) \ ,$$

so

$$v_{\pi(i,m')}(i,0) < v_{\pi(0,m')}(i,0) \ .$$

But

$$v_{\pi(i,m')}(i,0) = v_{\pi(0,m'-i)}(0,0)$$

$$\geq v_{\pi(0,m')}(0,0) \ ,$$

which implies that

$$v_{\pi(0,m')}(0,0) < v_{\pi(0,m')}(i,0) \ .$$

This is equivalent to

$$\sigma^{i} \cdot v_{\pi(0,m')}(0,0) < v_{\pi(0,m')}(0,0) \ ,$$

or

$$v_{\pi(0,m')}(0,0) > 0 .$$

This is a contradiction, since

$$v_{\pi(0,\infty)}(0,0) = 0 .$$

Thus

$$v_{\pi(0,m')}(i,1) \leq v_{\pi}(i,1), \quad \text{for} \quad 0 < i \leq m', \ \pi \in \mathcal{D}(\pi(0,\infty)) .$$

Let $\pi \in \mathcal{D}(\pi(0,m'))$ be the same policy as $\pi(0,m')$ except that it turns off the server in a state $(i,1)$, $i > m'$. Suppose that

$$v_{\pi}(i,1) < v_{\pi(0,m')}(i,1) .$$

By the policy improvement theorem,

$$v_{\pi(i,i)}(i,1) \leq v_{\pi}(i,1) ,$$

so

$$v_{\pi(i,i)}(i,1) < v_{\pi(0,m')}(i,1) .$$

Repeating the argument n times, we obtain

$$v_{\pi(ni,ni)}(ni,1) < v_{\pi(0,m')}(ni,1) .$$

Taking the limit on both sides as n tends to infinity, we obtain

$$v_{\pi(0,0)}(0,1) \leq \frac{K}{1-\omega} .$$

But

$$v_{\pi(0,0)}(0,1) = \frac{K}{1-\omega} + \frac{1-\xi}{1-X} B + \frac{XR_2 + R_1}{1-X} .$$

Since $B > 0$, $R_1 + R_2 \geq 0$ and $R_1 \geq 0$, we have a contradiction. Thus

$$v_{\pi(0,m')}(i,1) \leq v_\pi(i,1), \quad \text{for} \quad i > m', \; \pi \in \mathcal{D}(\pi(0,m')) .$$

We conclude that $\pi(0,m')$ is unimprovable and optimal.

Now let m' be the optimal m in $\pi(-1,m)$. From Section 3.2.3, we know that m' is finite. Suppose that $\pi(-1,m')$ is at least as good as $\pi(0,m)$ for $m \in N_0$. Then $\pi(-1,m')$ is unimprovable. This is shown in exactly the same way as the proof that $\pi(0,m')$ was unimprovable, so it will not be repeated. Thus $\pi(-1,m')$ is optimal.

We conclude that a natural hysteretic policy is optimal.

Q.E.D.

Lemma 5: Suppose $B > 0$, and let m' and m'' be the optimal values of m in $\pi(-1,m)$ and $\pi(0,m)$, respectively. If $m' < m''$, then $\pi(-1,m')$ is optimal. If $m'' < m'$, then $\pi(0,m'')$ is optimal.

Proof: Suppose first that $\pi(-1,m')$ is optimal. Then

$$v_{\pi(-1,m'')}(m'',1) \leq v_{\pi(0,m'')}(m'',1) .$$

This implies that

$$v_{\pi(-1,m'')}(m'',0) \leq v_{\pi(-1,m''+1)}(m'',0) ,$$

which in turn implies that $m' \leq m''$.

Suppose now that $\pi(0,m'')$ is optimal. Then

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$$v_{\pi(0,m')}(m',1) \leq v_{\pi(-1,m')}(m',1) \ .$$

This implies that

$$v_{\pi(0,m')}(m',0) \leq v_{\pi(0,m'+1)}(m',0) \ ,$$

which in turn implies that $m'' \leq m'$.

<div align="right">Q.E.D.</div>

Lemma 6: Suppose $B > 0$. Let m' and m'' be defined as in Lemma 5. Suppose that $m' = m''(= m)$. Let g be the function from N_0 into R, given by

$$g(i) = \frac{1-\sigma}{1-\sigma\psi} \cdot XD\psi^m + \frac{1-\psi}{1-\sigma\psi} \sigma C\sigma^{-m} - \xi B, \quad \text{for} \ i \in N_0 \ .$$

If $g(m) \leq 0$, then $\pi(0,m)$ is optimal. If $g(m) \geq 0$, then $\pi(-1,m)$ is optimal.

Proof: From Section 3.2.2, we have

$$v_{\pi(0,m)}(0,0) - v_{\pi(-1,m)}(0,0)$$

$$= \frac{AX(\sigma\psi)^m - \xi B\sigma^m}{1-X(\sigma\psi)^m} + \xi B\sigma^m - \frac{1-\sigma}{1-\sigma\psi} XD(\sigma\psi)^m$$

$$= (AX(\sigma\psi)^m - \xi B\sigma^m + \xi B\sigma^m(1-X(\sigma\psi)^m) - \frac{1-\sigma}{1-\sigma\psi} XD(\sigma\psi)^m(1-X(\sigma\psi)^m))$$

$$\cdot (1-X(\sigma\psi)^m)^{-1}$$

$$= (AX(\sigma\psi)^m - \xi BX(\sigma\psi)^m\sigma^m - \frac{1-\sigma}{1-\sigma\psi} XD(\sigma\psi)^m + \frac{1-\sigma}{1-\sigma\psi} X^2 D(\sigma\psi)^{2m})$$

$$\cdot (1-X(\sigma\psi)^m)^{-1}$$

$$= \frac{X(\sigma\psi)^m\sigma^m}{1-X(\sigma\psi)^m} ((A - \frac{1-\sigma}{1-\sigma\psi} D)\sigma^{-m} - \xi B + \frac{1-\sigma}{1-\sigma\psi} XD\psi^m)$$

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$$= \frac{X(\sigma\psi)^m \sigma^m}{1-X(\sigma\psi)^m} \left(\sigma \frac{1-\psi}{1-\sigma\psi} C\sigma^{-m} + \frac{1-\sigma}{1-\sigma\psi} XD\psi^m - \xi B \right)$$

$$= \frac{X(\sigma\psi)^m \sigma^m}{1-X(\sigma\psi)^m} \cdot g(m) ,$$

and the lemma follows.

<u>Theorem 7</u>: Let π^* denote the (an) optimal policy. If $B \leq 0$, then

$$\pi^* = \begin{cases} \pi(\infty,\infty) , & \text{for } A \leq 0 , \\ \pi(0,\infty) , & \text{for } A \geq 0, \ C \leq 0 , \\ \pi(-1,\infty), & \text{for } A \geq 0, \ C \geq 0 . \end{cases}$$

If $B > 0$, then

$$\pi^* = \begin{cases} \pi(-1,m') , & \text{for } m' < m'' , \\ \pi(0,m'') , & \text{for } m'' < m' , \end{cases}$$

$$\pi^* = \begin{Bmatrix} \pi(-1,m), & \text{for } g(m) \geq 0 \\ \pi(0,m) , & \text{for } g(m) \leq 0 \end{Bmatrix} \text{ for } m = m' = m'' ,$$

where

$$m' = \min\{m \in N_0 \mid m \geq \log(\frac{\xi B}{XD})/\log \psi\} ,$$

$$m'' = \min\{m \in N_0 \mid f(m) \geq \frac{XA}{\xi B}\} ,$$

and

$$f(m) = \frac{1-\sigma}{1-\sigma\psi} \psi^{-m} + X\sigma \frac{1-\psi}{1-\sigma\psi} \sigma^m ,$$

$$g(m) = \frac{1-\sigma}{1-\sigma\psi} XD\psi^m + \frac{1-\psi}{1-\sigma\psi} \sigma C\sigma^{-m} - \xi B .$$

<u>Proof</u>: These results follow directly from Lemmas 1, 2 and 3, Theorem 4, and Lemmas 5 and 6.

<u>Theorem 8</u>: Suppose $B > 0$, and let m' and m'' be defined as in the preceding theorem. Then

$$m' \begin{pmatrix} \leq \\ - \\ \geq \end{pmatrix} m''$$

if

$$\left(\frac{\xi}{X} \frac{B}{D}\right)^{\log \sigma / \log \psi} \begin{pmatrix} \leq \\ - \\ \geq \end{pmatrix} \frac{C}{\xi B} \ .$$

<u>Proof</u>: By definition,

$$m' = \min\{m \in \mathbb{N}_0 \,|\, \psi^m \leq \frac{\xi}{X} \frac{B}{D}\} \ ,$$

and

$$m'' = \min\{m \in \mathbb{N}_0 \,|\, f(m) \geq \frac{X}{\xi} \frac{A}{B} \ .$$

Since both $-\psi^m$ and $f(m)$ are increasing in m,

$$m' \begin{pmatrix} \leq \\ - \\ \geq \end{pmatrix} m''$$

is equivalent to

$$f\left(\log\left(\frac{\xi}{X} \frac{B}{D}\right) \log \psi\right) \begin{pmatrix} \leq \\ - \\ \geq \end{pmatrix} \frac{X}{\xi} \frac{A}{B} \ .$$

Simple algebra shows that this is equivalent to

$$\left(\frac{\xi}{X} \frac{B}{D}\right)^{\log \sigma / \log \psi} \begin{pmatrix} \leq \\ - \\ \geq \end{pmatrix} \frac{C}{\xi B} \ .$$

Q.E.D.

3.2.6 Bounds and Approximations.

Based on the results in the preceding sections, it should be easy to find the optimal policy now. The only problem may be to find m'' (given in Theorem 7). Since $f(m)$ is an increasing function of m,

m" may be found efficiently by the bisection method (see Wilde (1974, pp. 300-400)). To use this method, one needs an upper bound on m". This upper bound should be as small as possible. We will now give an upper bound which is also a good approximation to m" when certain conditions are met.

Notice that

$$f(m) \geq \frac{1-\sigma}{1-\sigma\psi} \cdot \psi^{-m}, \quad \text{for} \quad m \in N_0 \ ,$$

and that the expression on the right-hand side of the above inequality is also an increasing function of m. Therefore

$$m" \leq \min\{m \in N_0 | \frac{1-\sigma}{1-\sigma\psi} \psi^{-m} \geq \frac{\chi}{\xi} \frac{A}{B}\}$$

$$= \min\{m \in N_0 | m \geq \log(\frac{1-\sigma}{1-\sigma\psi} \frac{\xi}{\chi} \frac{B}{A})/\log \psi\} \ .$$

Letting

$$b = \log(\frac{1-\sigma}{1-\sigma\psi} \frac{\xi}{\chi} \frac{B}{A})/\log \psi \ ,$$

we obtain

$$m" \leq \min\{m \in N_0 | m \geq b\} \ .$$

For finding an optimal policy, it is more useful to have a relatively tight upper bound on $\min(m',m")$ instead. We obtain

$$\min(m',m") \leq \begin{cases} \min\{m \in N_0 | m \geq \log(\frac{\xi}{\chi} \frac{B}{D})/\log \psi\} \ , \\ \min\{m \in N_0 | m \geq b\} \end{cases}$$

or

$$\min(m',m") \leq \min\{m \in N_0 | m \geq \min(b,\log(\frac{\xi}{\chi} \frac{B}{D})/\log \psi)\} \ .$$

Now

$$b = \log(\frac{\xi}{\chi} \frac{B}{D})/\log \psi + \log(\frac{1-\sigma}{1-\sigma\psi} \frac{D}{A})/\log \psi \ ,$$

so

$$b < \log(\frac{\xi}{\chi} \frac{B}{D})/\log \psi$$

if and only if

$$\frac{1-\sigma}{1-\sigma\psi} \frac{D}{A} > 1 \ .$$

This is equivalent to $C < 0$. Therefore, b is a better upper bound on $\min(m',m'')$ than

$$\log(\frac{\xi}{\chi} \frac{B}{D})/\log \psi$$

if and only if $C < 0$.

The fact that b may also be a good approximation for m'' follows from the next theorem.

Theorem 9: If

$$1 - \chi(\sigma\psi)^b > \sigma \ ,$$

then m'' is either the smallest non-negative integer above b or the largest non-negative integer below b.

Proof: We only have to show that

$$f(b-1) < \frac{\chi A}{\xi B} \ .$$

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Now

$$f(b-1) = f(b) - (f(b) - f(b-1))$$

$$= f(b) - \frac{1-\psi}{1-\sigma\psi} (1-\sigma)(\psi^{-b} - X\sigma^b)$$

$$< f(b) - \frac{1-\psi}{1-\sigma\psi} \cdot X\sigma^{b+1} ,$$

since

$$1 - X(\sigma\psi)^b > \sigma .$$

Therefore

$$f(b-1) < f(b) - \frac{1-\psi}{1-\sigma\psi} X\sigma^{b+1}$$

$$= \frac{1-\sigma}{1-\sigma\psi} \cdot \psi^{-b}$$

$$= \frac{X}{\xi} \frac{A}{B} .$$

Q.E.D.

Corollary 10: If

$$\frac{A}{B} > \frac{\xi}{X} \frac{1-\sigma}{1-\sigma\psi} \left(\frac{X}{1-\sigma}\right)^{\log \psi/\log(\sigma\psi)} ,$$

then m'' is equal to the smallest non-negative integer above b or the largest non-negative integer below b.

Proof: Straightforward algebra shows that the condition of the Corollary is equivalent to the condition of the theorem.

Suppose that one has found b, and that it does not seem to be a good approximation to m''. Some graphs, indicating the true value of m'' as a function of X, ψ, σ and b, have been developed for this case. They can be found in Appendix B.

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3.2.7 The Case of Erlangian Service and Start-Up Times.

The Laplace transforms a, ξ, ψ and X may not always be easy to compute, given the cumulative distribution functions F and G. If the service times have k-Erlang distribution, then

$$\omega = \left(\frac{k\mu}{k\mu+\alpha}\right)^k ,$$

and

$$\psi = \left(\frac{k\mu}{k\mu+\alpha+\lambda-\lambda\psi}\right)^k = \left(\frac{\frac{k}{\rho}}{\frac{k}{\rho} + \frac{1}{\sigma} - \psi}\right)^k .$$

Since it is impossible to derive a closed form expression for ψ, some graphs, giving ψ as a function of κ, ρ and σ, have been developed. They can be found in Appendix C.

If the start-up times have a k-Erlang distribution, and if μ' denotes the start-up "rate," then

$$\xi = \left(\frac{k\mu'}{k\mu'+\alpha}\right)^k ,$$

and

$$X = \left(\frac{k\mu'}{k\mu'+\alpha+\lambda-\lambda\psi}\right)^k .$$

Having computed the values of ω, ψ, ξ and X, the optimal policy is easy to find.

4. The Case of Non-Decreasing Holding Cost Function.

The case where the holding cost function is an arbitrary non-decreasing function now will be investigated. Blackburn (1971) and Deb (1976) have also considered the problem where the holding cost

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function is not necessarily linear. The problem is considered both with and without discounting.

4.1 The Undiscounted Case.

In this section, costs are not discounted. Two optimality criteria are used, namely the average cost criterion and the undiscounted cost criterion. These criteria were described in Section 2. Recall that $\mathcal{D}$ is the set of deterministic stationary policies. Only these policies are considered here. Let $\mathcal{Y}$ denote the set of deterministic stationary policies which always turn the server on (or keep him on) at decision epochs where the number of customers in the system is larger than a certain number. It will be shown that only policies in $\mathcal{Y}$ need to be considered.

We assume that the service times are not instantaneous and that the holding cost function is not bounded from above. If desired, the analysis which follows can be extended so that these assumptions become unnecessary. Without loss of generality, we only allow policies which do not turn the server on and off repeatedly at the same point in time.

For each $\pi \in \mathcal{D}$, and $(i,j) \in N_0 \times \{0,1\}$, let $\varphi_\pi(i,j)$ denote the long run expected average cost, given that the start-state is (i,j) and that the policy π is used.

Lemma 11: For each $\pi \in \mathcal{D}$, there is a $\pi' \in \mathcal{Y}$ such that

$$\varphi_{\pi'}(i,j) \leq \varphi_\pi(i,j), \quad \text{for} \quad (i,j) \in N_0 \times \{0,1\} .$$

Proof: Let $\mathcal{F}$ denote the set of deterministic stationary policies which turn the server on if he is off and there are more than a certain number of customers in the system. Clearly

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$$\mathscr{Y} \subseteq \mathscr{T} \subseteq \mathscr{D} .$$

We prove the lemma by first showing that for each $\pi \in$, there is a $\pi' \in \mathscr{D}$ such that

$$\varphi_{\pi'}(i,j) \leq \varphi_{\pi}(i,j), \quad \text{for} \quad (i,j) \in \mathbb{N}_0 \times \{0,1\} ,$$

and then showing that for each $\pi \in \mathscr{T}$, there is a $\pi' \in \mathscr{Y}$ such that the above inequality holds again.

Therefore, consider a policy π in $\mathscr{D}$, but not in $\mathscr{T}$. Then there is a number, say k, such that π does not turn the server on if he is off and there are k or more customers in the system. This implies that

$$\varphi_{\pi}(i,0) = \varphi_{\pi}(j,0), \quad \text{for} \quad k \leq i \leq j ,$$

since the expected cost incurred until a state $(j,0)$ $(j \geq i)$ is reached, given that the start-state is $(i,0)$ $(i \geq k)$ and that the policy π is used, is finite.

Since h is a non-decreasing function, and since the number of customers in the system is always j or more, given that the start-state is j $(j \geq k)$ and that the policy π is used,

$$\varphi_{\pi}(j,0) \geq h(j), \quad \text{for} \quad j \geq k .$$

Together with the result above, this implies that

$$\varphi_{\pi}(i,0) = \infty, \quad \text{for} \quad i \geq k ,$$

since h is not bounded from above.

Let π' be the same policy as π except that it turns the server
on if he is off and there are k or more customers in the system.
Clearly

$$\varphi_{\pi'}(i,j) \begin{cases} \leq \varphi_{\pi}(i,j) = \infty, & \text{for } j = 0, \ i \geq k \ , \\ = \varphi_{\pi}(i,j) & , \ \text{otherwise} \ . \end{cases}$$

This completes the first part of the proof.

Now, consider a policy π in $\mathcal{F}$, but not in $\mathcal{Y}$. Then there
is a strictly increasing sequence of integers, $\{i_k\}_{k \in N}$, such that π
turns the server off at the decision epochs where he is on and the number
of customers in the system is i_k for some k in N. Since the service
times are not instantaneous, the probability that the number of customers
in the system will eventually exceed any given number is one. This
implies that the long run expected average holding cost, given any start-
state and the policy π, is equal to plus infinity, since for each
$k \in N$ the number of customers in the system cannot decrease below i_k
once it has been exceeded. Since the long run expected average cost
due to other costs than the holding cost is always larger than minus
infinity, we must have

$$\varphi_{\pi}(i,j) = \infty, \quad \text{for } (i,j) \in N_0 \times \{0,1\} \ .$$

Thus, any $\pi' \in \mathcal{Y}$ satisfies

$$\varphi_{\pi'}(i,j) \leq \varphi_{\pi}(i,j), \quad \text{for } (i,j) \in N_0 \times \{0,1\} \ .$$

This completes the second part of the proof.

Q.E.D.

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Lemma 12: For each $\pi \in \mathcal{Y}$, $\varphi_\pi(i,j)$ is constant over $(i,j) \in N_0 \times \{0,1\}$.

Proof: Assume that a policy in $\mathcal{Y}$, say π, is used, and let n be a number such that the server is always turned on (or kept on) when there are n or more customers in the system. Since the service times are not instantaneous, the probability that the number of customers in the system will eventually exceed n is one.

There are two mutually excluding and exhaustive cases, namely the case when the expected holding cost incurred during a service, given any number of customers in the system at the start of the service, is finite and the case when it is infinite. In the latter case, the long run expected average cost is equal to plus infinity for all start-states, and the lemma holds.

In the former case, the expected holding cost incurred during a service initiated with n or less number of customers in the system is bounded. Since the expected number of services given before the number of customers in the system exceeds n is bounded from above, this implies that the expected holding cost incurred until such a time is finite. Clearly, the expected service cost incurred until the number of customers in the system exceeds n is also finite.

The expected switching cost incurred until the number of customers in the system exceeds n can be seen to be finite as follows. There are two possible cases, the case where the start-up times are instantaneous and the case where these times are non-instantaneous. In the former case, the expected switching costs incurred until the number of customers in the system exceeds n is finite, since π cannot turn the server on and off repetitively at the same point in time and since the expected

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number of services given before the number of customers in the system
exceeds n is finite. In the latter case, the expected number of
turning the server on and off before the number of customers in the system
exceeds n is bounded. Therefore, the expected switching cost incurred
until the number of customers in the system exceeds n is finite.
Together with the previous results, this implies that the expected total
cost incurred until the number of customers exceed n is finite.

This in turn implies that $\varphi_\pi(i,j)$ is constant over
$(i,j) \in N_0 \times \{0,1\}$, since the states (i,j) are positive recurrent
for $i \geq n$ (recall that $\rho < 1$).

<div align="right">Q.E.D.</div>

Since we will only consider policies in $\mathscr{Y}$ hereafter, we will drop
the reference to the start-state in the following.

Theorem 13: There exists a natural hysteretic policy which is **average**
optimal.

Proof: Let π be a policy in $\mathscr{Y}$, and let n be least number such
that π keeps the server on if he is on and there are more than n
customers in the system. Let m be the least number greater than or
equal to n such that π turns the server on when the state of the
system is $(m,0)$. Then the policies π and $\pi(n,m)$ have the same
positive recurrent class (of states) and they take identical actions
within that class. Using Lemma 12, $\varphi_\pi = \varphi_{\pi(m,n)}$.

For each $i \in N$, let x_i denote the long run expected proportion
of time when there are i customers in the system given that the policy
$\pi(0,m-n)$ is used. Suppose that $n > 0$. Then the long run expected

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average holding cost is

$$\sum_{i \in N} x_i h(i) \quad , \quad \text{for} \quad \pi(0,m-n) \ ,$$

and

$$\sum_{i \in N} x_i h(i+n), \quad \text{for} \quad \pi(n,m) \ .$$

The long run expected average cost due to other costs than the holding costs are the same for $\pi(0,m-n)$ and $\pi(n,m)$. Since h is a non-decreasing function which is not bounded from above, and since each x_i is strictly positive,

$$\varphi_{\pi(0,m-n)} < \varphi_{\pi(n,m)} \ .$$

Thus, we can restrict our search for an optimal policy to the class of natural hysteretic policies. In order to prove that there is an average optimal natural hysteretic policy, we only need to show that there is a finite k such that

$$\varphi_{\pi(-1,0)} \leq \varphi_{\pi(0,m)}, \quad \text{for} \quad m \geq k \ .$$

This will now be shown.

For each i and m in N, let $t_{i,m}$ denote the long run expected proportion of time when there are i or more customers in the system, given that the policy $\pi(0,m)$ is used. Since $R_1 + R_2 \geq 0$,

$$\varphi_{\pi(0,m)} \geq t_{i,m} \cdot h(i) + \lambda \cdot \min(K,0), \quad \text{for} \quad i \in N, \ m \in N \ .$$

Choose $i \in N$ such that

$$\varphi_{\pi(-1,0)} < h(i) + \lambda \cdot \min(K,0) \ .$$

It can easily be shown that there is a k such that

$$t_{i,m} \geq (\varphi_{\pi(-1,0)} - \lambda \cdot \min(K,0))/h(i), \quad \text{for} \quad m \geq k \text{ ,}$$

since the right-hand side of the inequality is less than one. This implies that

$$\varphi_{\pi(-1,0)} \leq \varphi_{\pi(0,m)}, \quad \text{for} \quad m \geq n \text{ .}$$

<div align="right">Q.E.D.</div>

We now introduce some convenient terminology. If φ is the optimal long run expected average cost, then the <u>relative cost</u> incurred during a given time interval is the total cost incurred then minus φ times the length of the time interval.

For each $i \in N_0$, let C_m denote the cost incurred until the state $(i,1)$ is reached, and let f be the function from N_0 into R, given by

$$f(m) = E_{\pi(-1,m+1),(m,0)}\{C_m\} - E_{\pi(-1,m),(m,0)}\{C_m\}, \quad \text{for} \quad m \in N_0 \text{ .}$$

This function will play an important role in the following.

<u>Lemma 14</u>: If f is a non-decreasing function and φ denotes the optimal long run expected average cost, then the expected relative cost incurred until the server is on (regardless of the start-state) is minimized by $\pi(-1,m)$ (or equivalently by $\pi(0,m)$), where

$$m = \min\{m \in N_0 \mid f(m) \geq \varphi/\lambda(1-\rho)\} \text{ .}$$

Proof: If the system starts with the server on, the lemma is trivial. Therefore, assume now that the start-state is $(i,0)$ for some $i \in N_0$. With regard to the expected relative cost incurred until the surver is on, any policy which turns the server on eventually is equivalent to a policy $\pi(-1,m)$ (or $\pi(0,m)$) for some m.

For each $i \in N_0$, let T_i denote the time elapsed until the state $(i,1)$ is reached. We only have to show that

$$E_{\pi(-1,i+1),(i,0)}\{C_i - \varphi \cdot T_i\} - E_{\pi(-1,i),(i,0)}\{C_i - \varphi \cdot T_i\}$$

is non-negative for $i \geq m$ and non-positive for $i < m$. But this is just equivalent to

$$f(i) \begin{cases} \geq \varphi/\lambda(1-\rho), & \text{for} \quad i \geq m' \\ \leq \varphi/\lambda(1-\rho), & \text{for} \quad i < m' \, , \end{cases}$$

since

$$E_{\pi(-1,i+1),(i,0)}\{T_i\} - E_{\pi(-1,i),(i,0)}\{T_i\} = \frac{1}{\lambda(1-\rho)} \, .$$

Since f is a non-decreasing function, the lemma follows.

Q.E.D.

Lemma 15: For any set of real numbers a, b, c and d such that $b > 0$ and $d > 0$,

$$\frac{a}{b} \leq \frac{c}{d} \Longleftrightarrow \frac{a}{b} \leq \frac{a+c}{b+d} \Longleftrightarrow \frac{a+c}{b+d} \leq \frac{c}{d} \, .$$

Lemma 16: If f is a non-decreasing function, then the (an) average optimal value of m in $\pi(0,m)$ is given by

$$m = \min\{i \in N_0 \,|\, f(i) \geq \varphi_{\pi(0,i)}(\lambda(1-\rho)\} \, .$$

Proof: By renewal theory,

$$\varphi_{\pi(0,i)} = \{E_{\pi(0,m),(0,0)}\{C_0\} + R_2\}/\{E_{\pi(0,m),(0,0)}\{T_0\}\}, \quad \text{for} \quad i \in N_0 .$$

Since

$$E_{\pi(0,i+1),(0,0)}\{C_0\} = E_{\pi(0,i),(0,0)}\{C_0\} + f(i), \quad \text{for} \quad i \in N_0 ,$$

and since

$$E_{\pi(0,i+1),(0,0)}\{T_0\} = E_{\pi(0,i)}\{T_0\} = E_{\pi(0,i),(0,0)}\{T_1\} + \frac{1}{\lambda(1-\rho)} ,$$

$$\text{for} \quad i \in N_0 ,$$

we obtain

$$\varphi_{\pi(0,i+1)} = \{E_{\pi(0,i),(0,0)}\{C_0\} + R_2 + f(i)\}/\{E_{\pi(0,i),(0,0)}\{T_0\} + \frac{1}{\lambda(1-\rho)}\} ,$$

$$\text{for} \quad i \in N_0 .$$

Using Lemma 15,

$$\varphi_{\pi(0,i)} \overset{<}{=} \varphi_{\pi(0,i+1)}$$

if and only if

$$\varphi_{\pi(0,i)} \leq \lambda(1-\rho)f(i) .$$

By Theorem 13, there is an i such that

$$\varphi_{\pi(0,i)} \overset{<}{=} \varphi_{\pi(0,i+1)} ,$$

so m exists. By Lemma 15 and the definition of m,

$$\varphi_{\pi(0,m)} \overset{<}{=} \varphi_{\pi(0,m+1)} \leq \lambda(1-\rho)f(m) .$$

Since f is non-decreasing,

$$\varphi_{\pi(0,m+1)} \leq \lambda(1-\rho)f(m+1) \ .$$

Using Lemma 15 again, we obtain

$$\varphi_{\pi(0,m+1)} \leq \varphi_{\pi(0,m+2)} \leq \lambda(1-\rho)f(m+1) \ .$$

Continuing this procedure, we obtain

$$\varphi_{\pi(0,m)} \leq \varphi_{\pi(0,m+1)} \leq \cdots \ .$$

Now, since

$$\lambda(1-\rho)f(m-1) \leq \varphi_{\pi(0,m-1)} \ ,$$

we obtain (using Lemma 15)

$$\varphi_{\pi(0,m)} \leq \varphi_{\pi(0,m-1)} \ .$$

Since f is non-decreasing,

$$\lambda(1-\rho)f(m-2) \leq \varphi_{\pi(0,m-1)} \ .$$

Using Lemma 15 again, we obtain

$$\lambda(1-\rho)f(m-2) \leq \varphi_{\pi(0,m-2)} \ .$$

Continuing this procedure, we obtain

$$\varphi_{\pi(0,m)} \leq \varphi_{\pi(0,m-1)} \leq \cdots \ .$$

Thus,

$$\varphi_{\pi(0,m)} \leq \varphi_{\pi(0,i)}, \quad \text{for} \ \ i \in N_0 \ .$$

Q.E.D.

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Theorem 17: If f is non-decreasing, then there exists a natural hysteretic policy which is undiscounted optimal, and the optimal value of the upper intervention point (m) is given by

$$m = \min\{i \in N_0 \,|\, f(i) \geq \varphi/\lambda(1-\rho)\} \;,$$

where φ is the minimum long run expected average cost.

Proof: Consider the case where

$$\varphi_{\pi(-1,0)} < \varphi_{\pi(0,i)}, \quad \text{for } i \in N_0 \;.$$

In this case, only policies which eventually turn the server on and never turn him off are average optimal. By Lemma 14, the policy $\pi(-1,m)$ minimizes the long run expected relative cost for each start-state. This implies that $\pi(-1,m)$ is undiscounted optimal.

Consider the case where

$$\varphi_{\pi(0,m)} < \min\{\varphi_{\pi(-1,0)}, \varphi_{\pi(0,m-1)}, \varphi_{\pi(0,m+1)}\} \;.$$

From the proof of Theorem 13, only natural hysteretic policies can be average optimal. From the proof of Lemma 16, we have

$$\varphi_{\pi(0,m)} < \varphi_{\pi(0,i)}, \quad \text{for } i \in N_0 \;.$$

This implies that only policies which take the same actions as $\pi(0,m)$ for the states which are positive recurrent under $\pi(0,m)$ can be average optimal. Since $\pi(0,m)$ minimizes the expected relative cost until the server is on (for each start-state) by Lemma 14, $\pi(0,m)$ is undiscounted optimal.

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If $\varphi_{\pi(-1,m)} = \varphi_{\pi(0,m)}$ or $\varphi_{\pi(0,m)} = \varphi_{\pi(0,i)}$ for some $i \neq m$, then a finer analysis is needed to determine which of the corresponding policies is undiscounted optimal. The fact that one of the policies above is undiscounted optimal follows from Lemma 14. This completes the proof.

<div align="right">Q.E.D.</div>

Corollary 18: If the start-up times are zero, or if the holding cost function is convex, then there is a natural hysteretic policy which is undiscounted optimal, and the optimal value of the upper intervention point is given by Theorem 17.

Proof: We only have to show that f is a non-decreasing function. Consider the case where the start-up times are zero. In this case

$$f(i) = E_{\pi(0,i+1),(i,0)}\{C_i\} - R_1, \quad \text{for } i \in N_0 .$$

Clearly, f is non-decreasing, since h is non-decreasing.

Consider the case where the holding cost function is convex. Now

$$f(i) = E_{\pi(0,i+1),(i,0)}\{C_i\} - E_{\pi(0,i),(i,0)}\{C_i\}$$

$$= \frac{1}{\lambda} h(i) + E_{\pi(0,i+1),(i+1,1)}\{C_i\}$$

$$+ E_{\pi(0,i+1),(i+1,0)}\{C_{i+1}\} - E_{\pi(0,i),(i,0)}\{C_i\}, \quad \text{for } i \in N_0 .$$

The two first terms in the final right-hand side are non-decreasing in i, since h is non-decreasing. The difference between the two last terms is non-decreasing in i, since h is convex. Thus, f is non-decreasing.

<div align="right">Q.E.D.</div>

4.2 The Discounted Case.

The problem with discounting now will be considered. We assume that the start-up times are instantaneous. As before, let $\mathcal{D}$ denote the set of deterministic stationary policies and let $\mathcal{Y}$ denote the set of deterministic stationary policies which always turn the server on (or keep him on) at decision epochs where the number of customers in the system is larger than a certain number.

Without loss of generality, we use the convention that the server cannot be turned on immediately after he is turned off. The results obtained by Orkenyi (1976, Chapter 4) then are applicable. In particular, any unimprovable policy in $\mathcal{Y}$ is optimal. Also, the policy which always turns the server off (or keeps him off) is optimal if it is unimprovable and if its value function is finite-valued. These results will be used implicitly throughout the rest of this section.

A policy $\pi \in \mathcal{D}$ is unimprovable for the particular semi-Markov decision process under consideration here if

(a) $v_\pi(i,0) \leq v_{\pi'}(i,0)$,

(b) $v_\pi(i,0) \leq v_{\pi''}(i,0)$,

(c) $v_\pi(i,1) \leq v_{\pi'}(i,1)$,

(d) $v_\pi(i,1) \leq v_{\pi''}(i,1)$,

for $i \in N_0$, where π' and π'' are the same policies as π except that they respectively turn the server on (or keep him on) and turn him off (or keep him off) at the first decision epoch.

Let α, σ and ω be defined as before.

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Theorem 19: If

$$\frac{r}{\alpha} \geq R_2 \ ,$$

$$\overline{\sum_{i \in N}} \sigma^i \cdot h(i) < \infty \ ,$$

$$\sum_{j \in N_0} (\int_0^\infty F(t) \ \frac{(\lambda t)^j}{j!} \ e^{-(\lambda+\alpha)t} dt)(h(i+j)-h(i+j-1)) \leq K-(1-\omega)R_2 \ ,$$

$$\text{for} \ \ i \in N \ ,$$

then the policy which always turns the server off (or keeps him off)
is optimal.

Proof: We only need to show that $\pi(\infty,\infty)$ is unimprovable, since the
second condition of the theorem guarantees that its value function is
finite-valued.

Condition (a) holds for all $i \in N_0$, since $R_1 + R_2 \geq 0$. Condition
(b) and (d) hold trivially for all $i \in N_0$. It is now shown that con-
dition (c) also holds for all $i \in N_0$. Let π' be as in condition (c).
Then

$$v_{\pi'}(0,1) - v_{\pi(\infty,\infty)}(0,1) = \frac{r-\alpha R_2}{\lambda+\alpha}$$

$$\geq 0 \ .$$

Also

$$v_{\pi(\infty,\infty)}(i,1) = R_2 + \sum_{j \in N_0} (\int_0^\infty \frac{(\lambda t)^j}{j!} \ e^{-(\alpha+\lambda)t} dt) \cdot h(i+j), \ \ \text{for} \ \ i \in N \ ,$$

and

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$$v_{\pi'}(i,1) = K + \omega R_2 + \sum_{j \in N_0} \left(\int_0^\infty (1-F(t)) \frac{(\lambda t)^j}{j!} e^{-(\alpha+\lambda)t} dt \right) \cdot h(i+j)$$

$$+ \sum_{j \in N_0} \left(\int_0^\infty F(t) \frac{(\lambda t)^j}{j!} e^{-(\alpha+\lambda)t} dt \right) \cdot h(i+j-1) ,$$

$$\text{for } i \in N .$$

This implies that

$$v_{\pi'}(i,1) - v_{\pi(\infty,\infty)}(i,1) = K - (1-\omega)R_2$$

$$- \sum_{j \in N_0} \left(\int_0^\infty F(t) \frac{(\lambda t)^j}{j!} e^{-(\alpha+\lambda)t} dt \right)(h(i+j) - h(i+j-1)), \quad \text{for } i \in N .$$

Therefore,

$$v_{\pi(\infty,\infty)}(i,1) \leq v_{\pi'}(i,1), \quad \text{for } i \in N .$$

Thus condition (c) holds for all $i \in N_0$, and $\pi(\infty,\infty)$ is unimprovable.

$$\text{Q.E.D.}$$

<u>Corollary 20</u>: If $r \geq \alpha R_2$ and if

$$h(i+1) - h(i) \leq \frac{\alpha}{\omega} (K - (1-\omega)R_2), \quad \text{for } i \in N_0 ,$$

then $\pi(\infty,\infty)$ is optimal.

<u>Proof</u>: For each $i \in N$,

$$\sum_{j \in N_0} \left(\int_0^\infty F(t) \frac{(\lambda t)^j}{j!} e^{-(\alpha+\lambda)t} dt \right)(h(i+j) - h(i+j-1))$$

$$\leq \sum_{j \in N_0} \left(\int_0^\infty F(t) \frac{(\lambda t)^j}{j!} e^{-(\alpha+\lambda)t} dt \right) \cdot \frac{\alpha}{\omega} (K - (1-\omega)R_2)$$

$$= \left(\int_0^\infty F(t)e^{-\alpha t}dt \right) \cdot \frac{\alpha}{\omega} (K - (1-\omega)R_2)$$

$$= K - (1-\omega)R_2 \ .$$

Also

$$\sum_{i \in N} \sigma^i h(i) < \infty \ .$$

Thus the conditions of the theorem are satisfied and the corollary follows directly.

Q.E.D.

We will need to indicate the dependence of the value function of each policy on the start-up and shut-down costs. Therefore, for each $\pi \in \mathfrak{D}$, $a \in R$, $b \in R$, let $v_{\pi,a,b}$ denote the value function of policy π, given that the start-up cost is a and the shut-down cost is b.

For each $\pi \in \mathfrak{D}$, let u_π and w_π be the functions from $N_0 \times \{0,1\}$ into R defined by

$$u_\pi \equiv v_{\pi,R_1,-R_1} \ ,$$

and

$$w_\pi \equiv v_{\pi,-R_2,R_2} \ .$$

As will be seen later, these functions will be quite useful in the following.

Lemma 21: If π is a policy (in $\mathfrak{D}$) which always turns the server on (or keeps him on) when the number of customers in the system is greater than or equal to, say m, and if in addition,

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$$v_\pi(m,1) \leq u_{\pi(m,m+1)}(m,1) \; ,$$

$$u_{\pi(i-1,i)}(i,1) \leq u_{\pi(i,i+1)}(i,1), \quad \text{for} \quad i > m \; ,$$

then π satisfies the conditions (a), (b), (c) and (d) for $i \geq m$.

Proof: The conditions (a) and (c) are trivially satisfied for $i \geq m$.
We now show that condition (b) is satisfied for $i \geq m$.

Observe that condition (b) is equivalent to

$$v_\pi(i,1) \leq u_{\pi(i,i+1)}(i,1) \; ,$$

for $i \geq m$. We prove that

$$v_\pi(i,1) \leq u_{\pi(i,i+1)}(i,1), \quad \text{for} \quad i \geq m \; ,$$

by induction on i. The above inequality holds trivially for $i = m$.
Suppose that it has been proven to hold for some $i \geq m$. Then

$$v_\pi(i,1) \leq u_{\pi(i,i+1)}(i,1) \; ,$$

or equivalently

$$v_\pi(i+1,1) \leq u_{\pi(i,i+1)}(i+1,1) \; .$$

Using this together with the last assumption of the lemma, we obtain

$$v_\pi(i+1,1) \leq u_{\pi(i,i+1)}(i+1,1)$$

$$\leq u_{\pi(i+1,i+2)}(i+1,1) \; .$$

This completes the induction proof, and condition (b) is satisfied for
$i \geq m$.

That condition (d) holds for $i \geq m$, is seen as follows. From the above results,

$$v_\pi(i,0) = v_\pi(i,1) + R_1$$

$$\leq v_{\pi''}(i,0)$$

$$= v_{\pi''}(i,1) - R_2, \quad \text{for} \quad i \geq m .$$

This implies that

$$v_\pi(i,1) \leq v_{\pi''}(i,1) - (R_1 + R_2)$$

$$\leq v_{\pi''}(i,1), \quad \text{for} \quad i \geq m ,$$

since $R_1 + R_2 \geq 0$. This completes the proof of the lemma.

Q.E.D.

Let m' and m'' be the smallest numbers in $N \cup \{\infty\}$ such that

$$v_{\pi(-1,m')}(0,0) \leq v_{\pi(-1,m)}(0,0), \quad \text{for} \quad m \in N_0 ,$$

and

$$v_{\pi(0,m'')}(0,0) \leq v_{\pi(0,m)}(0,0), \quad \text{for} \quad m \in N_0 .$$

That m' and m'' exist, follows from the fact that

$$\lim_{m \to \infty} v_{\pi(-1,m)}(0,0) = v_{\pi(-1,\infty)}(0,0) ,$$

and

$$\lim_{m \to \infty} v_{\pi(0,m)}(0,0) = v_{\pi(0,\infty)}(0,0) .$$

Also notice that

$$v_{\pi(-1,1)}(0,0) \leq v_{\pi(-1,0)}(0,0) ,$$

since

$$v_{\pi(-1,0)}(0,0) - v_{\pi(-1,1)}(0,0) = \frac{r+\alpha R_1}{\lambda+\alpha}$$

$$\geq 0 \; .$$

Lemma 22: If π is a policy (in $\mathcal{Y}$) which always turns the server on (or keeps him on) when the number of customers in the system is greater than or equal to, say $m(m > 0)$, and if in addition,

$$v_{\pi'}(m-1,0) \leq \min\{v_{\pi}(m-1,0),\; v_{\pi''}(m-1,0)\} \; ,$$

where π' and π'' are the same policies as π with the only exception that they do not turn the server on in the states $(m,0)$ and $(m+1,0)$, respectively, then

$$u_{\pi(m-1,m)}(m,1) \leq v_{\pi}(m,1) \leq u_{\pi(m,m+1)}(m,1) \; .$$

Proof: Clearly

$$v_{\pi'}(m-1,0) \leq v_{\pi}(m-1,0)$$

is equivalent to

$$u_{\pi(m-1,m)}(m,1) \leq v_{\pi'}(m,1) \; ,$$

and

$$v_{\pi'}(m-1,0) \leq v_{\pi''}(m-1,0)$$

is equivalent to

$$v_{\pi'}(m,1) \leq u_{\pi(m,m+1)}(m,1) \; .$$

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Combining these results, the lemma follows.

<div align="right">Q.E.D.</div>

Let f and g be the two functions from N_0 into R given by

$$f(m) = u_{\pi(m,m+1)}(m,1) - v_{\pi(-1,m)}(m,1), \quad \text{for} \quad m \in N_0 ,$$

and

$$g(m) = u_{\pi(m,m+1)}(m,1) - v_{\pi(0,m)}(m,1) , \quad \text{for} \quad m \in N_0 .$$

Lemma 23: If there is a k such that

$$u_{\pi(m-1,m)}(m,1) - u_{\pi(m,m+1)}(m,1) \begin{cases} \geq 0, & \text{for} \quad i < k , \\ \leq 0, & \text{for} \quad i \geq k , \end{cases}$$

then the conditions (a), (b), (c) and (d) are satisfied for $i \geq m'$ and $i \geq m''$ for $\pi = \pi(-1,m')$ and $\pi = \pi(0,m'')$, respectively.

Proof: By Lemma 22,

$$v_{\pi(m-1,m)}(m,1) \leq u_{\pi(m,m+1)}(m,1), \quad \text{for} \quad m = m' \quad \text{and} \quad m = m'' .$$

Using this together with the condition of the lemma, we obtain

$$u_{\pi(m-1,m)}(m,1) \leq u_{\pi(m,m+1)}(m,1), \quad \text{for} \quad m \geq \min(m',m'') .$$

By Lemma 22,

$$f(m') \geq 0 ,$$

and

$$g(m'') \geq 0 .$$

Thus, we can use Lemma 21 to obtain that the conditions (a), (b), (c)

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and (d) are satisfied for $i \geq m'$ and $i \geq m''$ for $\pi = \pi(-1,m')$ and $\pi(0,m'')$, respectively.

<div align="right">Q.E.D.</div>

Lemma 24: Under the condition of Lemma 23, the conditions (a) and (b) are satisfied for all $i \in N_0$ for both $\pi = \pi(-1,m')$ and $\pi = \pi(0,m'')$.

Proof: Follows directly from Lemma 23 and the definition of m' and m''.

Lemma 25: Under the condition of Lemma 23,

$$f(m) \begin{cases} \leq 0, & \text{for } m < m' \\ \geq 0, & \text{for } m \geq m' \end{cases},$$

$$g(m) \begin{cases} \leq 0, & \text{for } m < m'' \\ \geq 0, & \text{for } m \geq m'' \end{cases},$$

and m' and m'' are the smallest integers in N satisfying the above inequalities.

Proof: Since condition (b) holds for $i \geq m'$ and $i \geq m''$ for $\pi = \pi(-1,m')$ and $\pi = \pi(0,m'')$, respectively,

$$f(m) \geq 0, \quad \text{for } m \geq m' ,$$

and

$$g(m) \geq 0, \quad \text{for } m \geq m'' .$$

We prove that

$$f(m) \leq 0, \quad \text{for } m < m' ,$$

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by induction on m. Clearly

$$f(0) \leq 0 \; ,$$

since

$$v_{\pi(-1,0)}(0,0) - v_{\pi(-1,1)}(0,0) = \frac{r + \alpha R_1}{\lambda + \alpha}$$

$$\geq 0 \; .$$

Suppose that we have proven that

$$f(m) \leq 0, \quad \text{for some} \quad m < m' - 1 \; .$$

This is equivalent to

$$u_{\pi(m,m+1)}(m+1,1) \leq v_{\pi(-1,m+1)}(m+1,1) \; .$$

But by the condition of the lemma,

$$u_{\pi(m,m+1)}(m+1,1) \geq u_{\pi(m+1,m+2)}(m+1,1) \quad (m < m'-1) \; .$$

Thus

$$u_{\pi(m+1,m+2)}(m+1,1) \leq v_{\pi(-1,m+1)}(m+1,1) \; ,$$

or equivalently

$$f(m+1) \leq 0 \; .$$

This completes the induction proof.

That

$$g(m) \leq 0, \quad \text{for} \quad m < m'' \; ,$$

can be shown in a quite similar manner. That

$$g(0) \le 0 \;,$$

follows from the fact that $R_1 + R_2 \ge 0$. Suppose that we have proven that

$$g(m) \le 0, \quad \text{for some} \quad m < m'' - 1 \;.$$

This implies that

$$u_{\pi(m,m+1)}(m+1,1) \le v_{\pi(0,m+1)}(m+1,1) \;.$$

But by the condition of the lemma,

$$u_{\pi(m,m+1)}(m+1,1) \ge u_{\pi(m+1,m+2)}(m+1,1) \quad (m < m''-1) \;.$$

Thus

$$u_{\pi(m+1,m+2)}(m+1,1) \le v_{\pi(0,m+1)}(m+1,1) \;,$$

or equivalently

$$g(m+1) \le 0 \;.$$

This completes the induction proof. The last assertion of the lemma follows trivially from the definition of m' and m''.

<div align="right">Q.E.D.</div>

Lemma 26: If

$$v_{\pi(-1,m')}(i,1) \le v_{\pi(i,m')}(i,1), \quad \text{for} \quad i < m' \;,$$

and there is a k such that

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$$u_{\pi(i-1,i)}(i,1) - u_{\pi(i,i+1)}(i,1) \begin{cases} \geq 0 & \text{for } i < k \\ \leq 0 & \text{for } i \geq k, \end{cases}$$

then $\pi(-1,m')$ is optimal.

Proof: By Lemmas 23 and 24, we only need to show that condition (d) holds for $i < m'$ for $\pi = \pi(-1,m')$. But this follows directly from the first condition of the lemma.

<div align="right">Q.E.D.</div>

Lemma 27: If

$$w_{\pi(i-1,i)}(i,1) \leq w_{\pi(i,i+1)}(i,1), \quad \text{for } i \in N,$$

then

$$v_{\pi(i-1,m)}(i,1) \geq v_{\pi(i,m)}(i,1), \quad \text{for } 0 < i < m.$$

Proof: It is enough to prove that

$$w_{\pi(i-1,i)}(i,1) \leq v_{\pi(i,m'')}(i,1), \quad \text{for } 0 < i < m,$$

since this implies that

$$v_{\pi(i-1,m)}(i,1) \leq v_{\pi(i,m)}(i,1), \quad \text{for } 0 < i < m.$$

We use an induction proof. Clearly

$$w_{\pi(i,i+1)}(i,1) \leq v_{\pi(i,m)}(i,1), \quad \text{for } i = m - 1,$$

since $R_1 + R_2 \geq 0$. Using the condition of the lemma for $i = m-1$, we obtain

$$w_{\pi(i-1,i)}(i,1) \leq v_{\pi(i,m)}(i,1), \quad \text{for } i = m - 1.$$

Suppose that we have proven that

$$w_{\pi(i-1,i)}(i,1) \leq v_{\pi(i,m)}(i,1), \quad \text{for some} \quad 1 < i < m .$$

This implies that

$$w_{\pi(i-1,i)}(i,1) \leq v_{\pi(i-1,m)}(i,1) ,$$

which is equivalent to

$$w_{\pi(i-1,i)}(i-1,1) \leq v_{\pi(i-1,m)}(i-1,1) .$$

Using the condition of the lemma, we obtain

$$w_{\pi(i-2,i-1)}(i-1,1) \leq v_{\pi(i-1,m)}(i-1,1) .$$

This completes the induction proof.

Q.E.D.

Lemma 28: If

$$v_{\pi(0,m'')}(0,1) \leq v_{\pi(-1,m'')}(0,1) ,$$

$$w_{\pi(i-1,i)}(i,1) \leq w_{\pi(i,i+1)}(i,1), \quad \text{for} \quad i \in N ,$$

and if there is a k such that

$$u_{\pi(i-1,i)}(i,1) - u_{\pi(i,i+1)}(i,1) \begin{cases} \geq 0, & \text{for} \quad i < k , \\ \leq 0, & \text{for} \quad i \geq k , \end{cases}$$

then $\pi(0,m'')$ is optimal.

Proof: By Lemmas 23 and 24, we only need to show that conditions (c) and (d) hold for $i < m''$.

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Condition (c) holds trivially for $i > 0$. It also holds for $i = 0$ by the first assumption of the lemma. Condition (d) holds for $i < m''$ by Lemma 27. Thus, $\pi(0,m'')$ is unimprovable and optimal.

$$\text{Q.E.D.}$$

Theorem 29: If m'' is finite, if

$$w_{\pi(i-1,i)}(i,1) \leq w_{\pi(i,i+1)}(i,1), \quad \text{for} \quad i \in N ,$$

and if there is a k such that

$$u_{\pi(i-1,i)}(i,1) - u_{\pi(i,i+1)}(i,1) \begin{cases} \geq 0, & \text{for} \quad i < k , \\ \leq 0, & \text{for} \quad i \geq k , \end{cases}$$

then there is a natural hysteretic policy which is optimal, and it has the following characterization.

If $m' < m''$, then $\pi(-1,m')$ is optimal. If $m'' < m'$, then $\pi(0,m'')$ is optimal. If $m' = m''$, then $\pi(-1,m')$ or $\pi(0,m'')$ is optimal according to which of the two policies minimizes $v_\pi(0,0)$.

Proof: Consider the policy $\pi(0,m'')$. If

$$v_{\pi(0,m'')}(0,1) \leq v_{\pi(-1,m'')}(0,1) ,$$

then $\pi(0,m'')$ is optimal by Lemma 28. If

$$v_{\pi(0,m'')}(0,1) > v_{\pi(-1,m'')}(0,1) ,$$

then m' is finite and $\pi(-1,m')$ is optimal. We now prove this assertion.

Therefore, assume now that

$$v_{\pi(0,m')}(0,1) > v_{\pi(-1,m'')}(0,1) \ .$$

This is equivalent to

$$v_{\pi(0,m'')}(m'',1) > v_{\pi(-1,m'')}(m'',1) \ .$$

By Lemma 25,

$$m' = \min\{m \in N \cup\{\infty\} \,|\, u_{\pi(m,m+1)}(m,1) \geq v_{\pi(-1,m)}(m,1)\}$$

and

$$m'' = \min\{m \in N \,|\, u_{\pi(m,m+1)}(m,1) \geq v_{\pi(0,m)}(m,1)\} \ .$$

Thus m' is less than or equal to m'', and thus it is finite.

By Lemma 27,

$$v_{\pi(0,m')}(i,1) \leq v_{\pi(i,m')}(i,1), \quad \text{for} \quad 0 < i < m' \ .$$

This leads to

$$v_{\pi(-1,m')}(i,1) \leq v_{\pi(-1,m'')}(i,1)$$

$$\leq v_{\pi(0,m'')}(i,1)$$

$$\leq v_{\pi(0,m')}(i,1)$$

$$\leq v_{\pi(i,m')}(i,1), \quad \text{for} \quad 0 < i < m' \ .$$

Thus, the conditions of Lemma 26 are satisfied, and we can conclude that $\pi(-1,m')$ is optimal.

Therefore, if $\pi(-1,m')$ is optimal, then $m' \leq m''$. Suppose that $\pi(-1,m')$ is not optimal. Then

$$v_{\pi(i,m')}(m',1) \leq v_{\pi(-1,m')}(m',1), \quad \text{for some} \quad 0 \leq i < m' \, .$$

Using Lemma 27, we obtain

$$v_{\pi(0,m')}(m',1) \leq v_{\pi(-1,m')}(m',1) \, .$$

Therefore, m'' is less than or equal to m' by Lemma 25. This completes the proof of the theorem.

Q.E.D.

Lemma 30: If there is an $\epsilon > 0$ such that

$$h(i+1) - h(i) \geq \frac{\alpha}{\omega} (K + (1-\omega)R_1) + \epsilon, \quad \text{for} \quad i \in N_0 \, ,$$

then m'' is finite.

Proof: Suppose first that

$$h(i+1) - h(i) = \frac{\alpha}{\omega} (K + (1-\omega)R_1) + \epsilon, \quad \text{for} \quad i \in N_0 \, ,$$

and let m''_0 denote the value of m'' for this case. From Section 3.2, we know that m''_0 is finite.

Consider now the general case where

$$h(i+1) - h(i) \geq \frac{\alpha}{\omega} (K + (1-\omega)R_1) + \epsilon, \quad \text{for} \quad i \in N_0 \, .$$

Clearly

$$u_{\pi(m''_0,m''_0+1)}(m''_0,1) \geq v_{\pi(0,m''_0)}(m''_0,1) \, ,$$

since the number of customers in the system is always at least as large when $\pi(m,m+1)$ is used as when $\pi(0,m)$ is used (for each m).

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If

$$h(i+1) - h(i) = \frac{\alpha}{\omega}(K + (1-\omega)R_1), \quad \text{for } i \in N_0 \,,$$

then

$$u_{\pi(i-1,i)}(i,1) - u_{\pi(i,i+1)}(i,1) = 0, \quad \text{for } i \in N \,.$$

Therefore, in the general case,

$$u_{\pi(i-1,i)}(i,1) - u_{\pi(i,i+1)}(i,1) \geq 0, \quad \text{for } i \in N \,,$$

since the number of customers in the system is always at least as large when $\pi(i,i+1)$ is used as when $\pi(i-1,i)$ is used (for each $i \in N$).

Therefore, we can use Lemma 25 to conclude that m'' is less than or equal to m_0''. Thus, m'' is finite.

<div align="right">Q.E.D.</div>

Lemma 31: If there is an $\epsilon > 0$ and an $n < \infty$ such that

$$h(i+1) - h(i) \geq \frac{\alpha}{\omega}(K + (1-\omega)R_1) + \epsilon, \quad \text{for } i \geq n \,,$$

if

$$w_{\pi(i-1,i)}(i,1) \leq w_{\pi(i,i+1)}(i,1), \quad \text{for } i \in N \,,$$

and if there is a k such that

$$u_{\pi(i-1,i)}(i,1) - u_{\pi(i,i+1)}(i,1) \begin{cases} \geq 0, & \text{for } i < k \,, \\ \leq 0, & \text{for } i \geq k \,, \end{cases}$$

then m'' is finite.

Proof: Let m_1' be the smallest integer such that

$$v_{\pi(r,m_1'')}(k,0) \leq v_{\pi(n,m)}(k,0), \quad \text{for} \quad m > n \ .$$

Consider now the queueing system where the holding cost function $h(i)$ has been replaced by the holding cost function $h(i+n)$. Let m_o'' be as in Lemma 30 for this system. Then m_o'' is finite, and

$$m_o'' - m_1'' - n \ .$$

This implies that m_1'' is finite.

By Lemma 27,

$$v_{\pi(0,m_1'')}(n,1) \leq v_{\pi(n,m_1'')}(n,1) \ .$$

This implies that

$$u_{\pi(m_1'',m_1''+1)}(m_1'',1) \geq v_{\pi(n,m_1'')}(m_1'',1)$$

$$\geq v_{\pi(0,m_1'')}(m_1'',1) \ .$$

Using Lemma 25, we conclude that $m'' \leq m_1''$. Thus m'' is finite.

Q.E.D.

Theorem 32: If h is convex, if

$$h(i+1) - h(i) \geq \frac{\alpha}{\omega} (K - (1-\omega)R_2), \quad \text{for} \quad i \in N_0 \ ,$$

and if

$$h(i+1) - h(i) > \frac{\alpha}{\omega} (K + (1-\omega)R_1), \quad \text{for some} \quad i \in N_0 \ ,$$

then the condition of Theorem 29 hold and there is a natural hysteretic policy which is optimal.

Proof: It is easy to see that

$$u_{\pi(i,i+1)}(i,1) - u_{\pi(i-1,i)}(i,1)$$

is a non-decreasing function of i, since h is convex. Thus the last condition of Theorem 29 holds.

If

$$h(i+1) - h(i) = \frac{\alpha}{\omega}\,(K - (1-\omega)R_2), \quad \text{for} \quad i \in N_0 ,$$

then

$$w_{\pi(i-1,i)}(i,1) = w_{\pi(i,i+1)}(i,1), \quad \text{for} \quad i \in N .$$

Therefore,

$$h(i+1) - h(i) \geq \frac{\alpha}{\omega}\,(K - (1-\omega)R_2), \quad \text{for} \quad i \in N_0 ,$$

implies that the second condition of Theorem 29 is satisfied.

By Lemma 31, the first condition $(m'' < \infty)$ of Theorem 29 also holds. Thus, the theorem follows.

<div align="right">Q.E.D.</div>

Theorem 33: If there is an $\epsilon > 0$ such that

$$h(i+1) - h(i) \geq \frac{\alpha}{\omega}\,(K + (1-\omega)R_1) + \epsilon, \quad \text{for} \quad i \in N_0 ,$$

the conditions of Theorem 29 are satisfied and there is a natural hysteretic policy which is optimal.

Proof: Since $R_1 + R_2 \geq 0$,

$$h(i+1) - h(i) \geq \frac{\alpha}{\omega}\,(K - (1-\omega)R_2), \quad \text{for} \quad i \in N_0 .$$

As it was shown in the proof of Theorem 32, this implies that the second condition of Theorem 29 holds.

As it was shown in Lemma 30,

$$h(i+1) - h(i) \geq \frac{\alpha}{\omega} (K - (1-\omega)R_1), \quad \text{for} \quad i \in N_0 ,$$

implies that the last condition of Theorem 29 holds. By the same lemma, the first condition $(m'' < \infty)$ of Theorem 29 holds. Thus, the theorem follows.

<div align="right">Q.E.D.</div>

Blackburn (1971) studied the case where the holding cost function is convex. His Theorem 14 (in Chapter 3) is equivalent to our Theorem 33 with the exception that we do not require the holding cost function to be convex. When we do require that the holding cost function is convex, in Theorem 32, the other conditions are made considerably weaker.

Deb (1976) obtained a result almost identical to our Theorem 33 in an independent study of a related problem.

We will now give an efficient algorithm for computing m' and m'' when they are finite. The only requirement is that $f(m)$ and $g(m)$ can be calculated efficiently for various values of m. For the cases where the holding cost function is a quadratic or exponential function, closed form expressions can be obtained for f and g. The same is true for the case where the service times are constant or have an Erlang distribution. If the holding cost function has a linear tail, nearly closed form expressions can be obtained for f and g. In the following, we assume that f and g can be evaluated efficiently at all relevant points. Since the algorithm is the same for both m' and m'', we shall only outline it for m'.

Therefore, suppose m' is finite. The following algorithm finds

m'.

Step 1: Let m = 1.

Step 2: While f(m) < 0, double m.

Step 3: Let $\underline{m}$ = m/2 and $\overline{m}$ = m.

Step 4: While $\overline{m} - \underline{m} > 1$, do the following:

Let m = $(\overline{m} + \underline{m})/2$.

If f(m) $\geq$ 0, then set $\overline{m}$ = m.

Otherwise, set $\underline{m}$ = m.

Step 5: Let m' = $\overline{m}$.

That this algorithm really finds m' follows from Lemma 25. The algorithm
is essentially a bisection method, and it requires only $2 \log_2 m'$ function
evaluations.

We close this section by mentioning that most of the results in
this section can be easily extended to cover hysteretic policies which
are not necessarily natural. In particular, an algorithm, similar to the
one above, can be constructed so that it finds the optimal value of both
n and m in $\pi(n,m)$, where n is not necessarily less than or equal
to zero.

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APPENDIX A: Definition of Basic Symbols

α = interest rate.

$\gamma = \displaystyle\int_0^\infty t^2 dF(t)$ = second moment of the service time.

$\zeta = \displaystyle\int_0^\infty t\, dG(t)$ = expected length of the start-up time.

$\eta = \displaystyle\int_0^\infty t^2 dG(t)$ = second moment of the start-up time.

λ = arrival rate.

$\mu = \left(\displaystyle\int_0^\infty t\, dF(t) \right)^{-1}$ = service rate.

$\xi = \displaystyle\int_0^\infty e^{-\alpha t} dG(t)$ = Laplace transform of the start-up time.

$\rho = \dfrac{\lambda}{\mu}$ = load on system.

$\sigma = \dfrac{\alpha}{\lambda + \alpha}$ = Laplace transform of the inter-arrival times.

$\chi = \displaystyle\int_0^\infty e^{-(\alpha + \lambda - \lambda\psi)t} dG(t).$

$\psi = \displaystyle\int_0^\infty e^{-(\alpha + \lambda - \lambda\psi)t} dF(t)$ = Laplace transform of busy period.

$\omega = \displaystyle\int_0^\infty e^{-\alpha t} dF(t)$ = Laplace transform of the service time.

F = cumulative distribution function for the service times.

G = cumulative distribution function for the start-up times.

APPENDIX B: The Optimal Value of m in $\pi(0,m)$

This appendix contains graphs which show the optimal value of m in $\pi(0,m)$ for various values of the system parameters χ, ψ, σ and b. The optimal value of m in $\pi(0,m)$ is given by

$$m = \min\{i \in N_0 | f(i) \geq \frac{1-\sigma}{1-\sigma\psi} \cdot \psi^{-b}\} \ ,$$

where

$$f(i) = \frac{1-\sigma}{1-\sigma\psi} \ \psi^{-i} + \chi \cdot \frac{1-\psi}{1-\sigma\psi} \ \sigma^{i+1}, \quad \text{for} \quad i \in N_0 \ .$$

It should be noted that a logarithmic scale has been used for σ in the graphs.

X = 1
Ψ = .1

X = 1
Ψ = .5

84

$X = 1$

$\psi = .7$

$X = .1$

$\psi = .8$

σ

X = 1

ψ = .9

m=0

m=2

m=3

σ

X = 1

ψ = .99

m=0

m=2

m=3

86

σ

.98

m=0

.96

X = .6

.9

Ψ = .5

.8

.6

m=2 m=3

1   3   5   7   9   11   13   b

σ

.98

m=0

.96

X = .6

.9

Ψ = .7

.8

.6

m=2 m=3

1   3   5   7   9   11   13   b

X = .6
Ψ = .8

X = .6
Ψ = .9

88

X = .6
Ψ = .99

X = .4
Ψ = .7

89

X = .4
Ψ = .8

X = .4
Ψ = .9

σ

.98

.96

X = .4

Ψ = .99

.9

.8

.6

m=0

m=2

m=3

1 3 5 7 9 11 13 b

σ

.98

.96

X = .2

Ψ = .8

.9

.8

.6

m=0

m=2

m=3

1 3 5 7 9 11 13 b

X = .2
Ψ = .9

X = .2
Ψ = .99

92

σ

.98

.96

X = .1

ψ = .9

.9

.8

.6

m=0

m=1

m=2

1   3   5   7   9   11   13   b

σ

.98

.96

X = .1

ψ = .99

.9

.8

.6

m=0

m=1

m=2

1   3   5   7   9   11   13   b

93
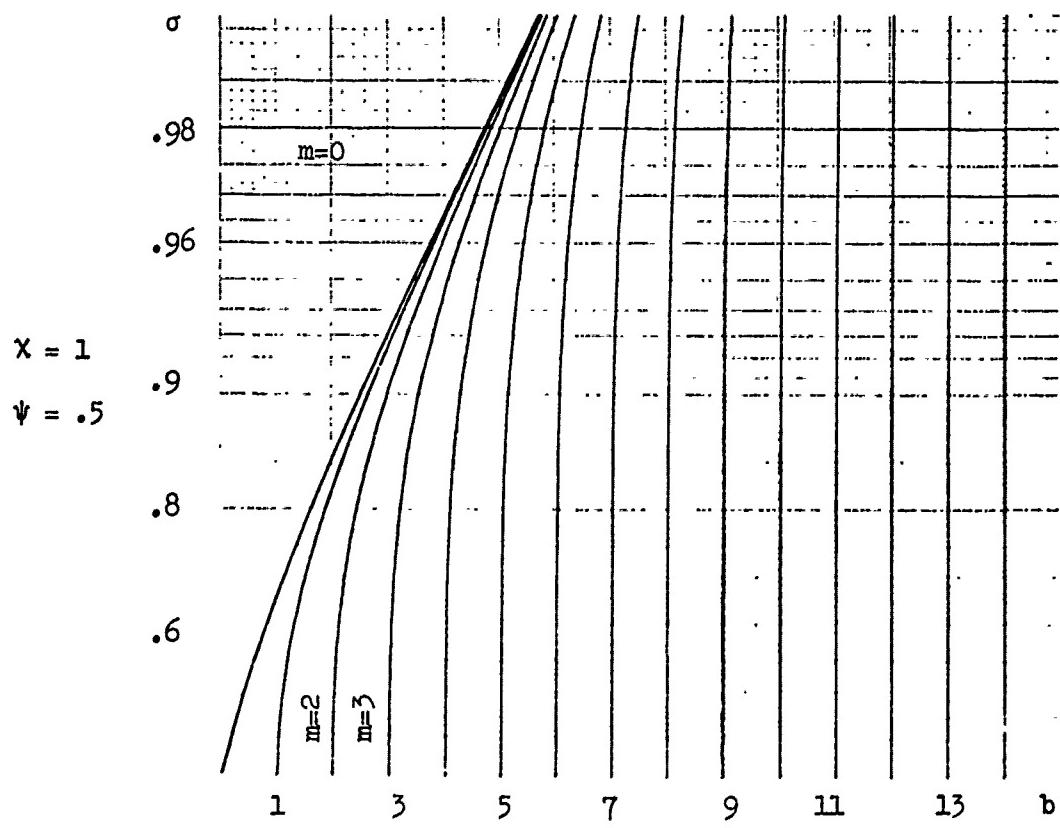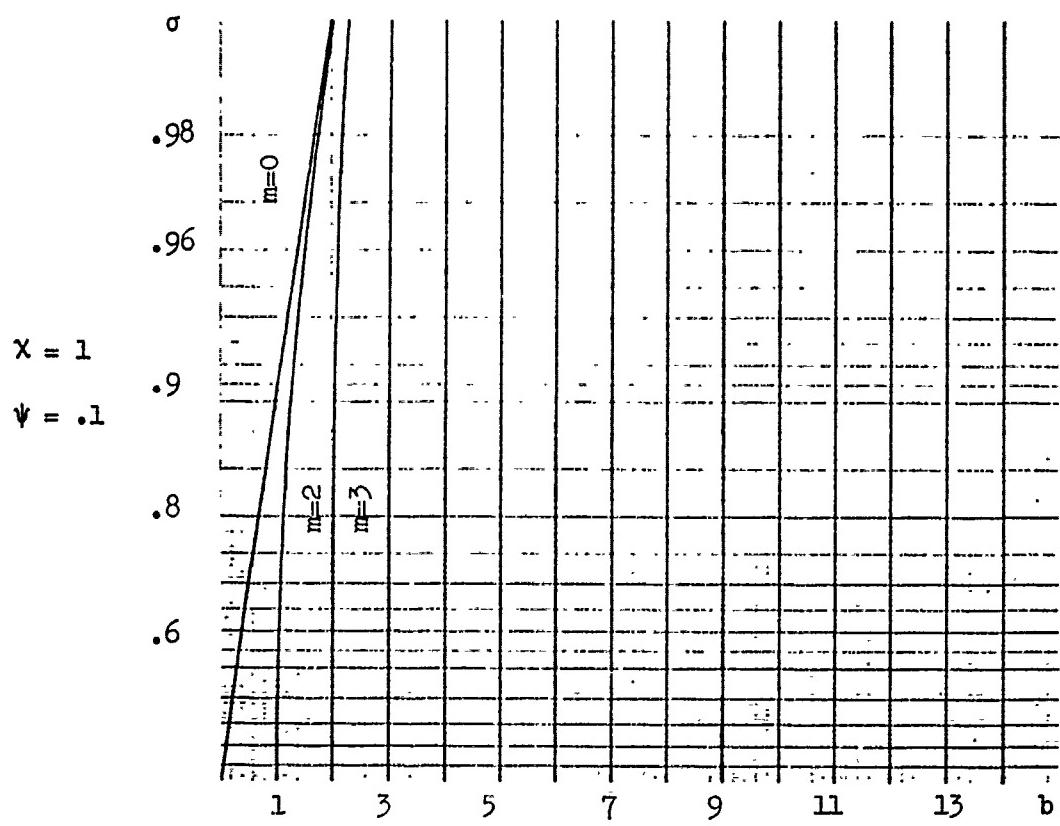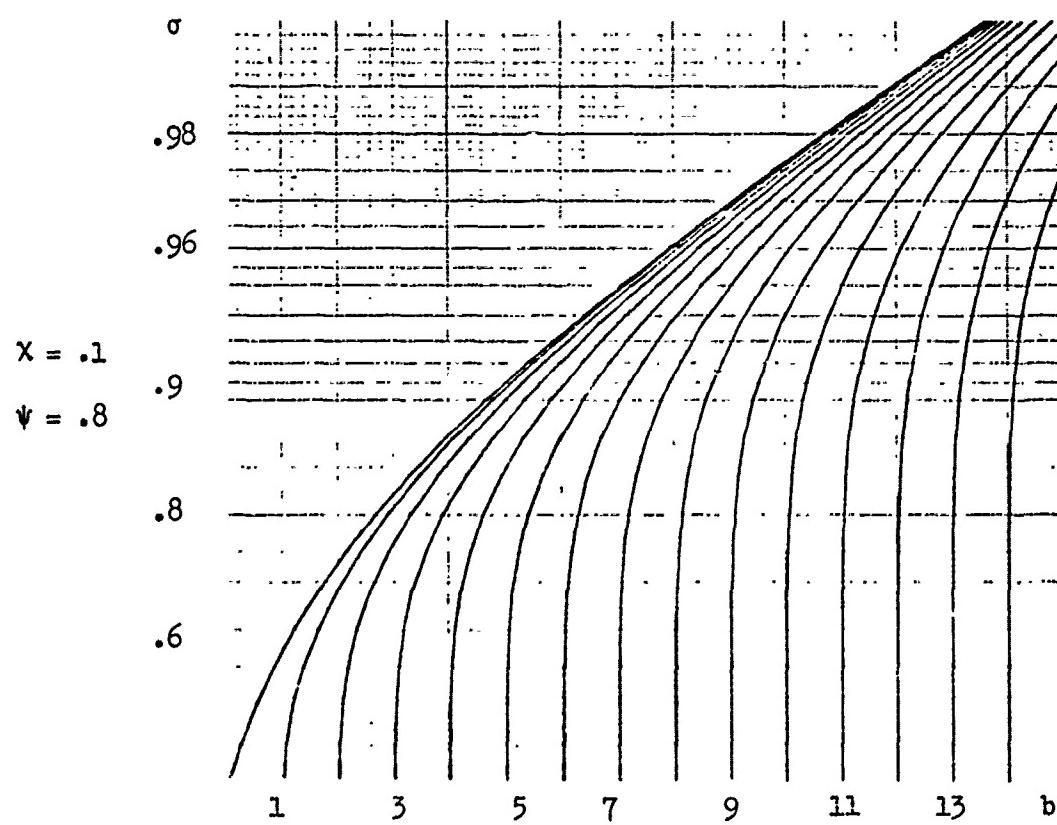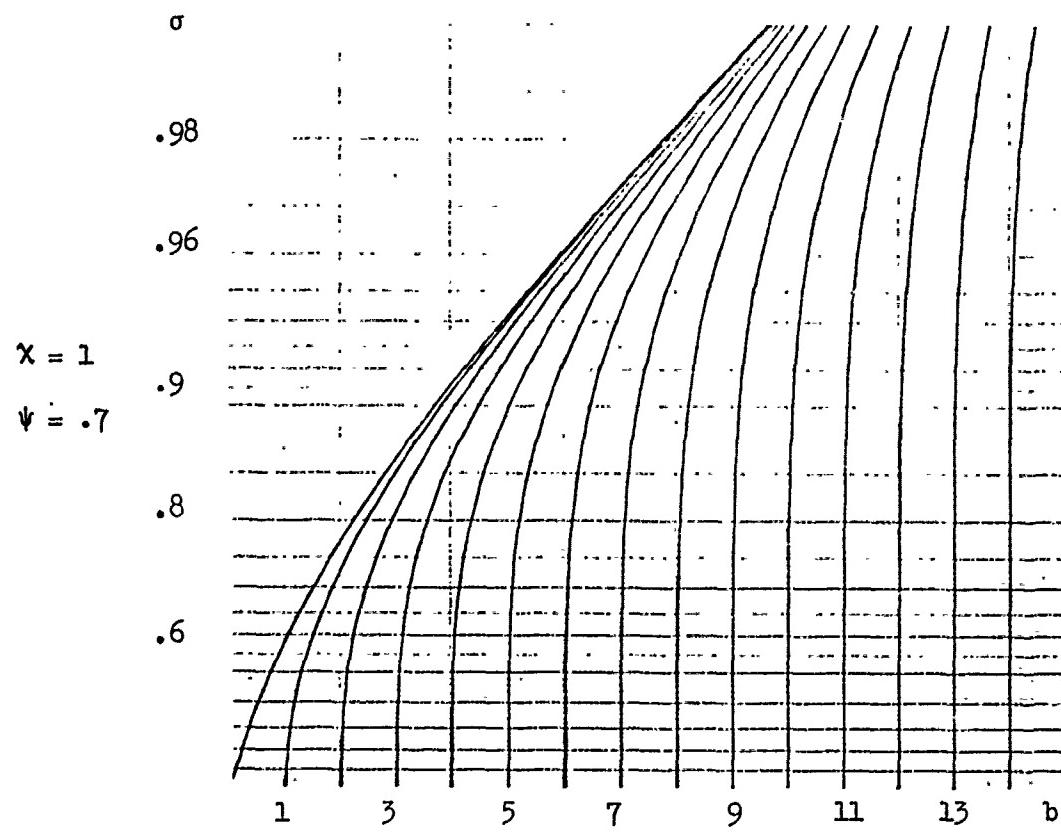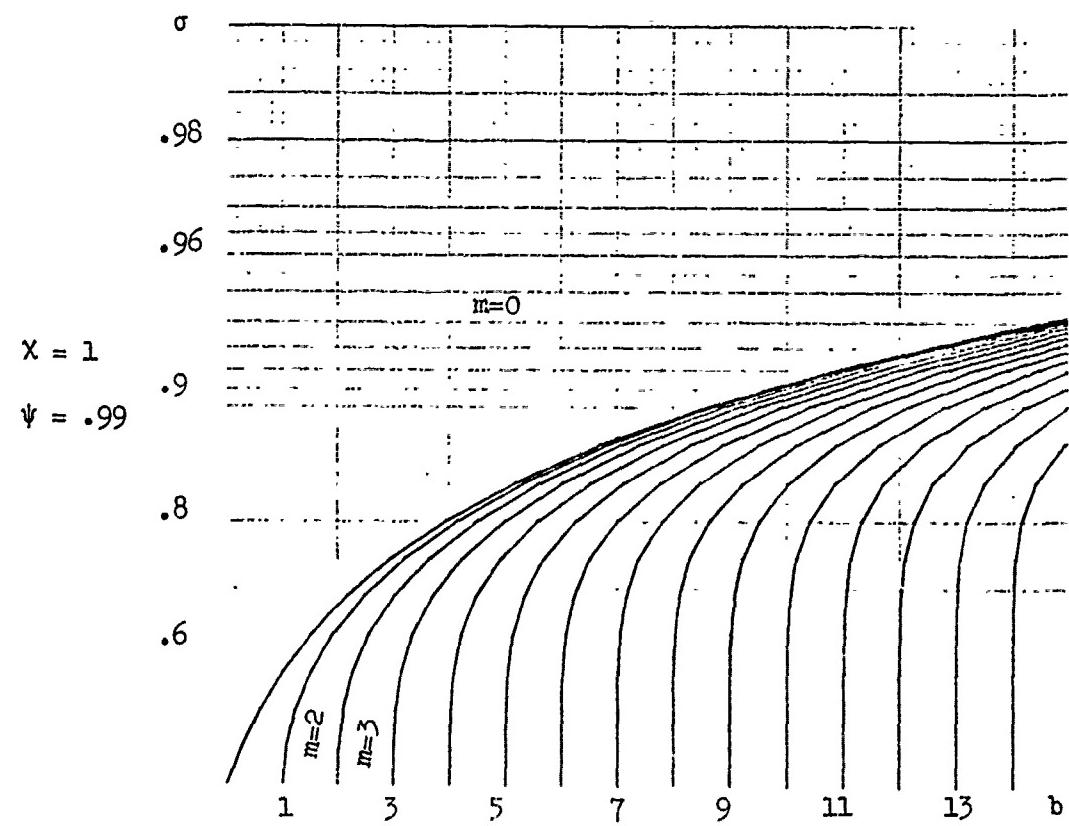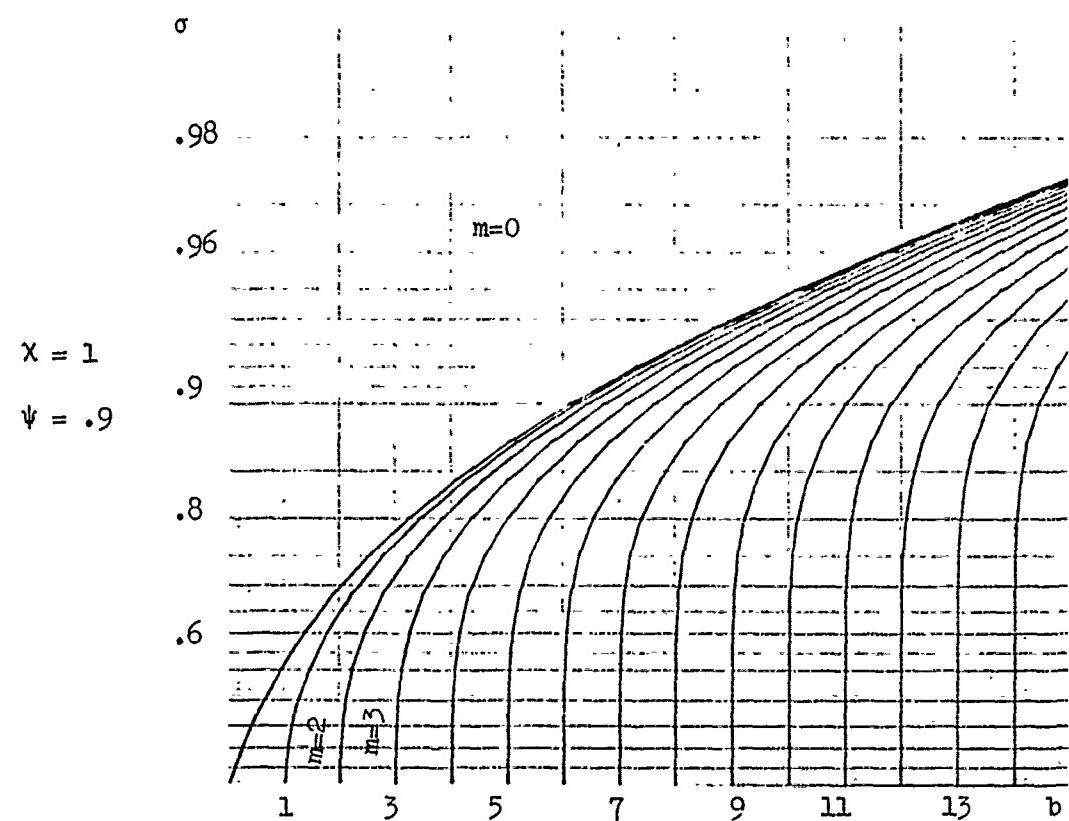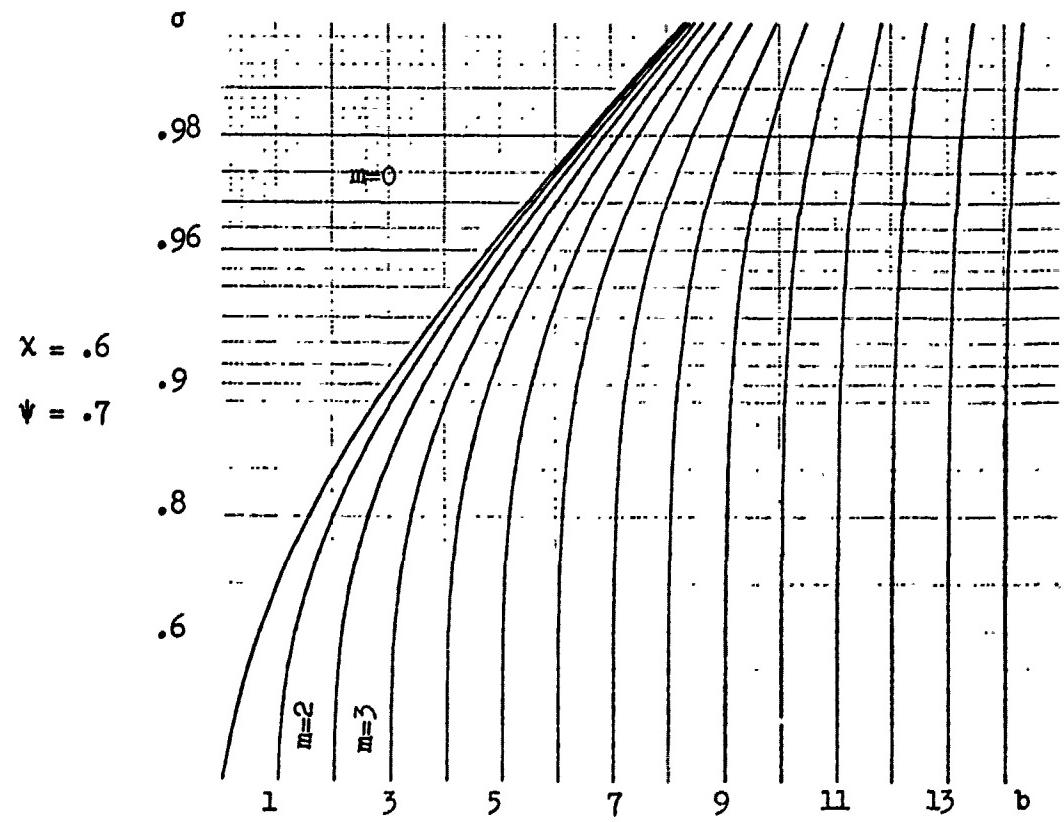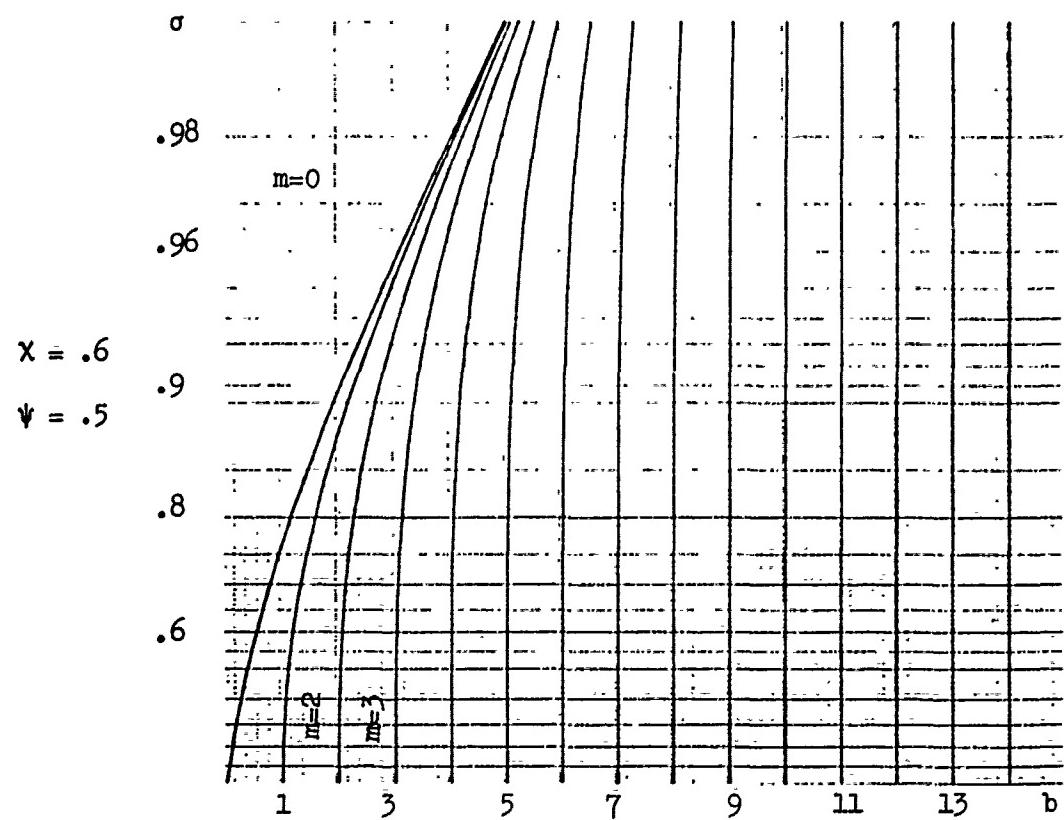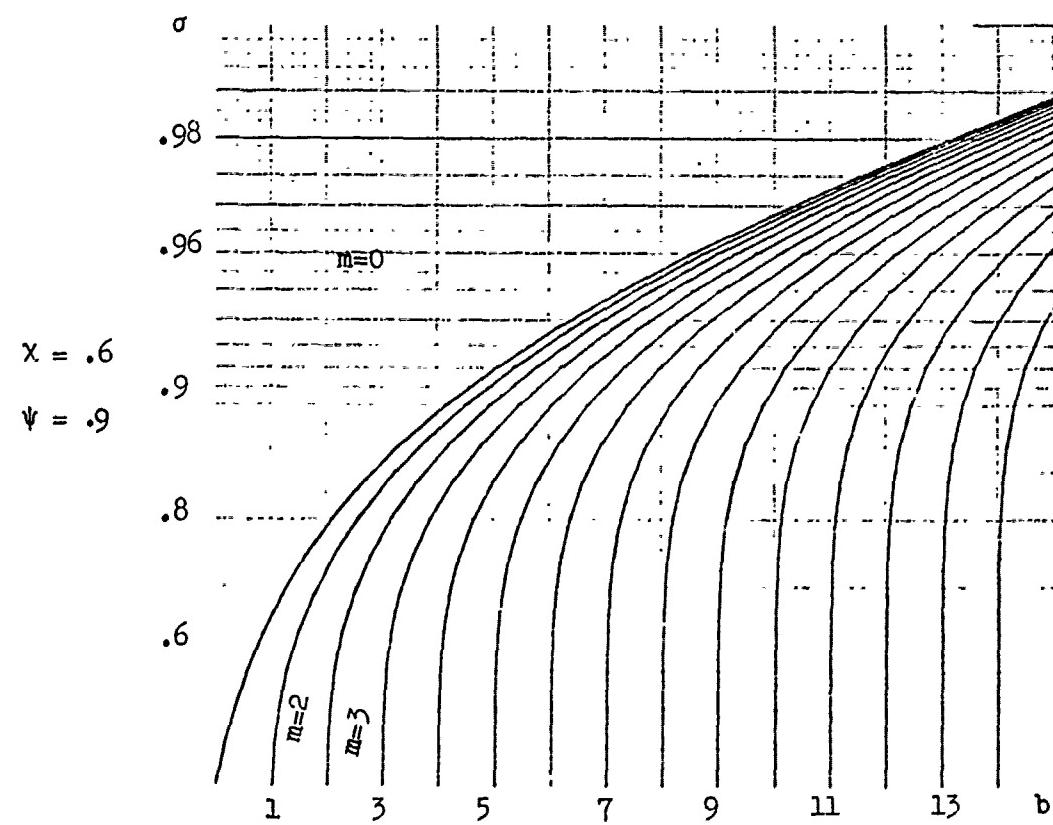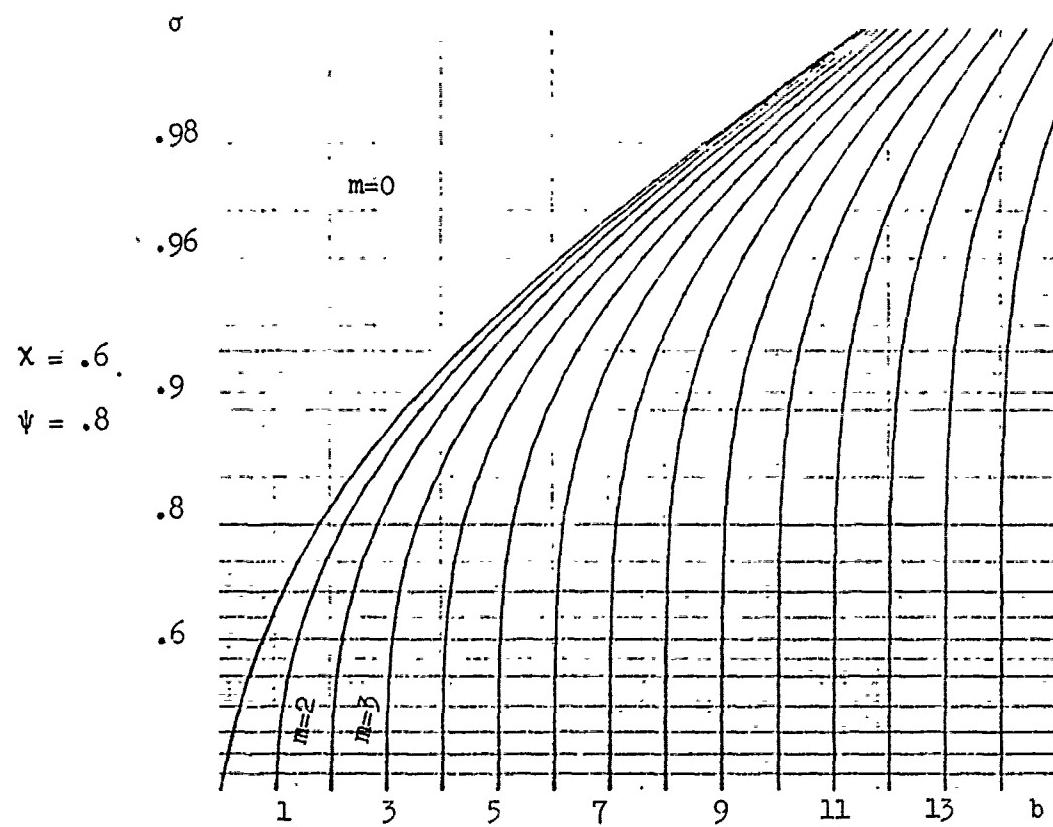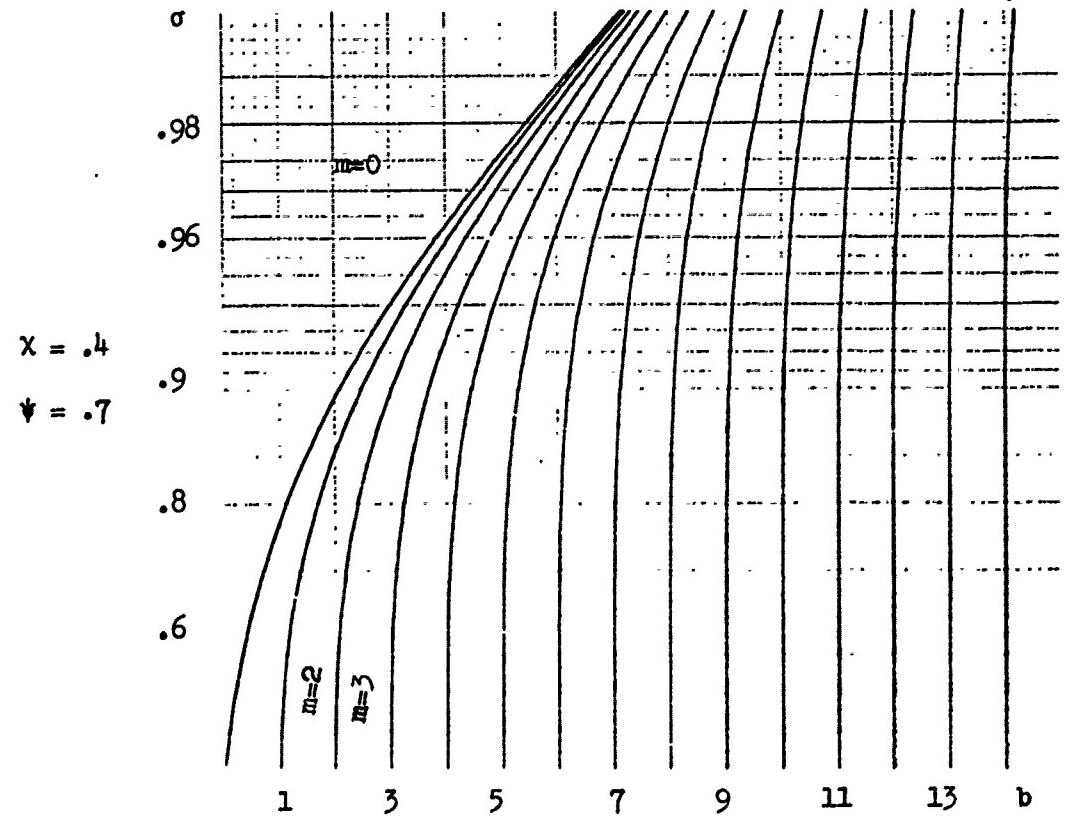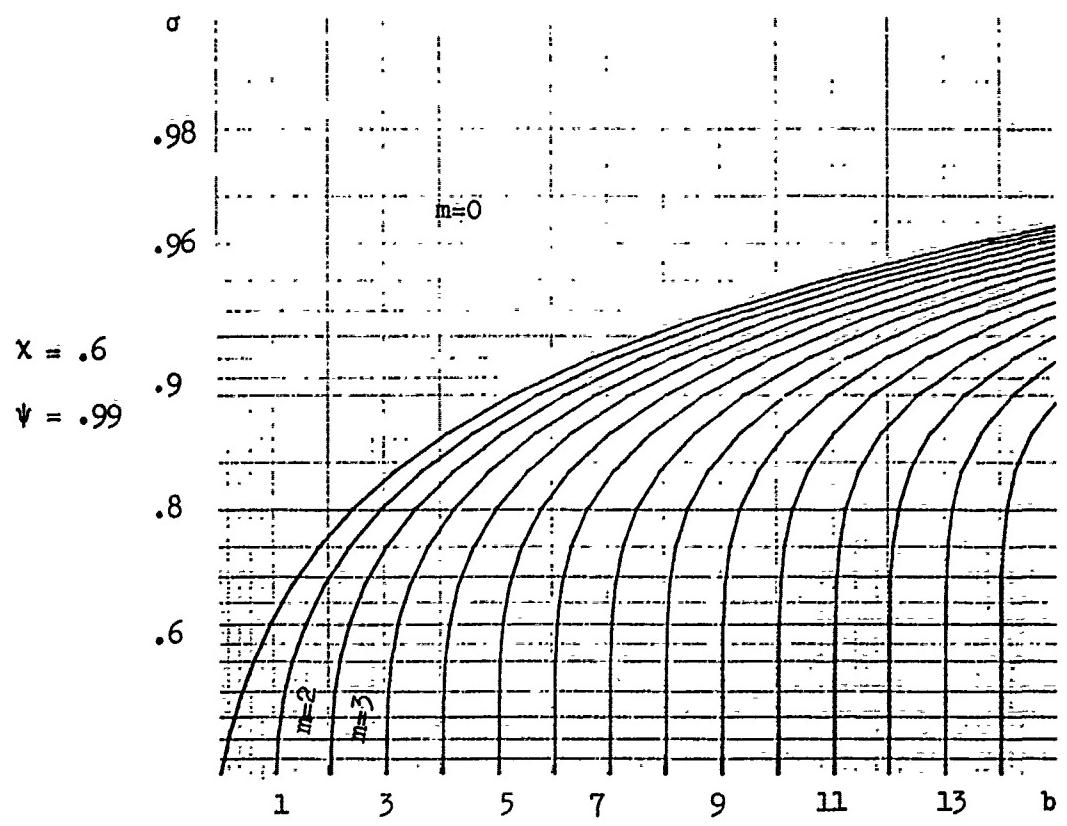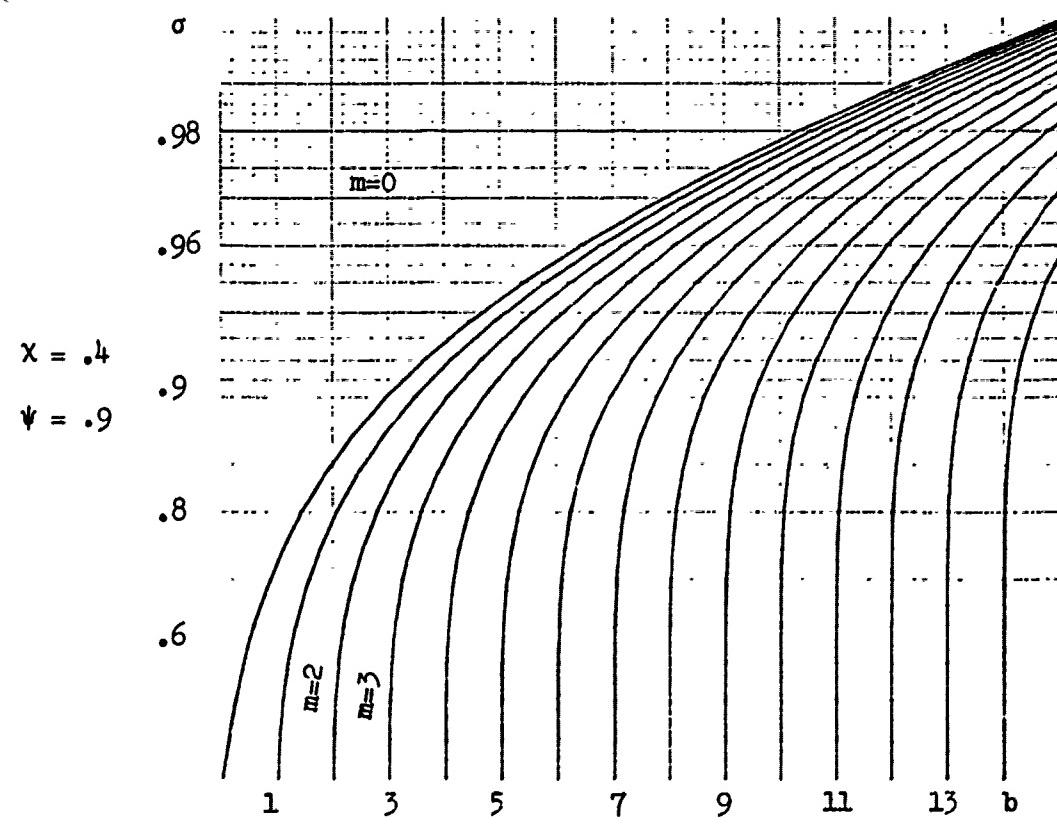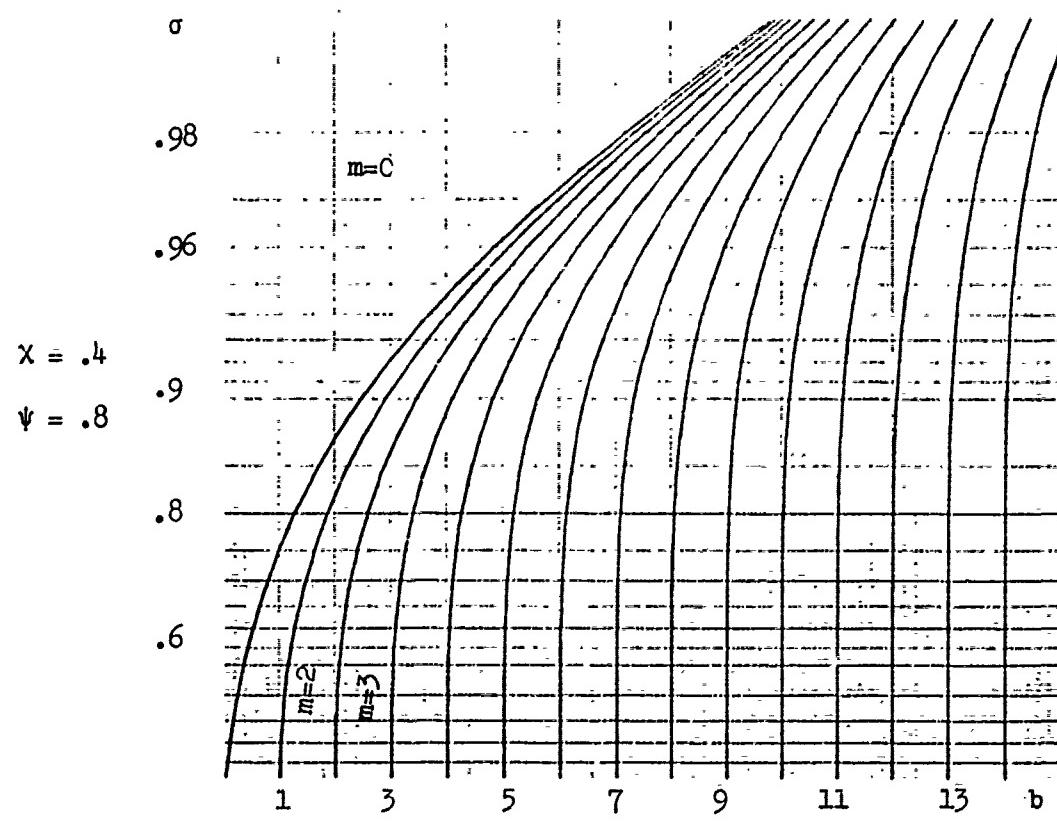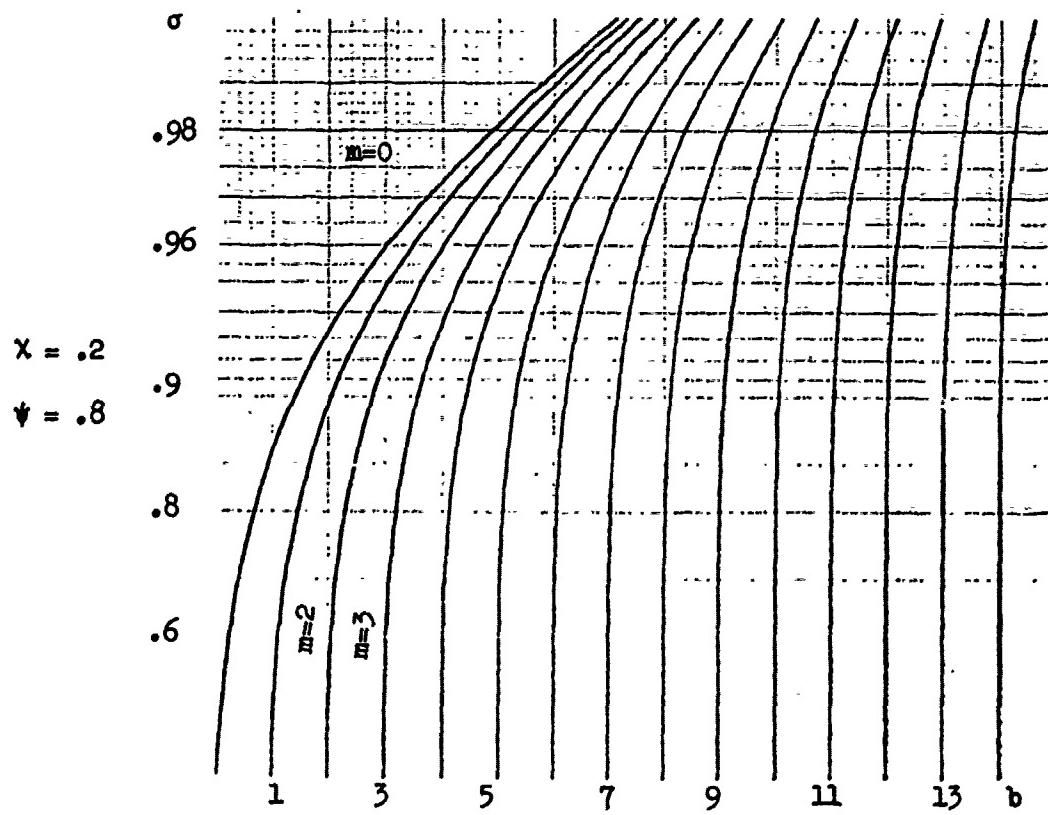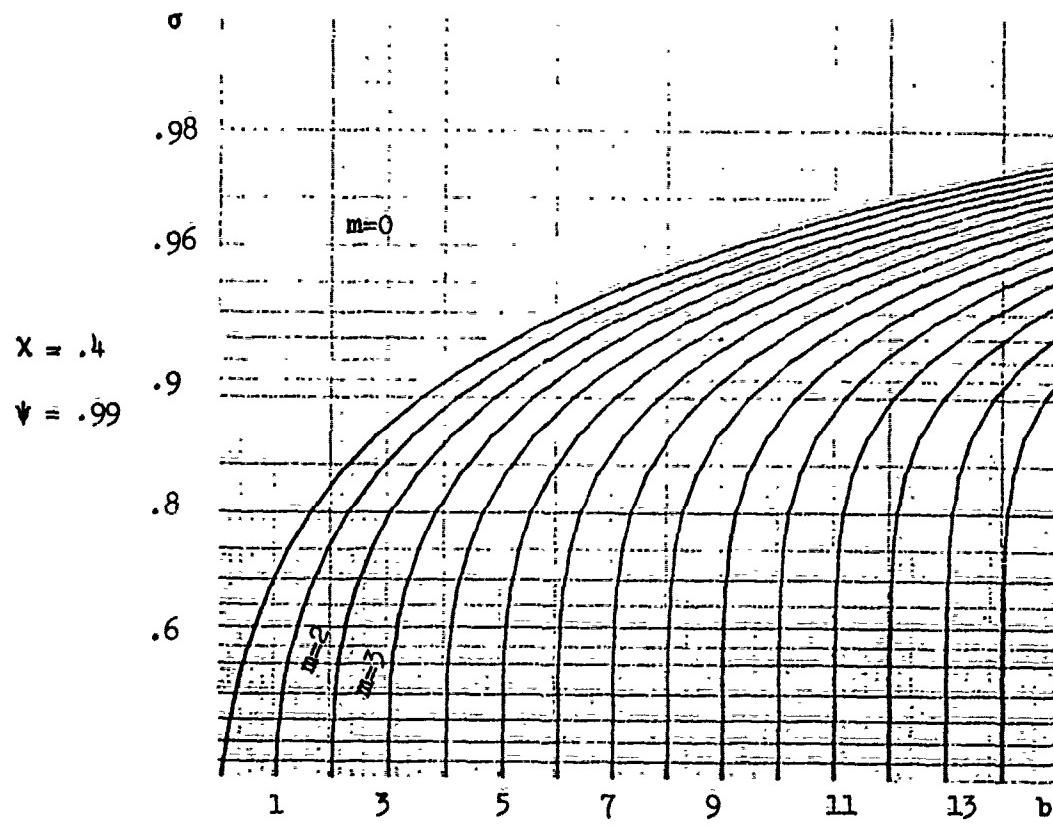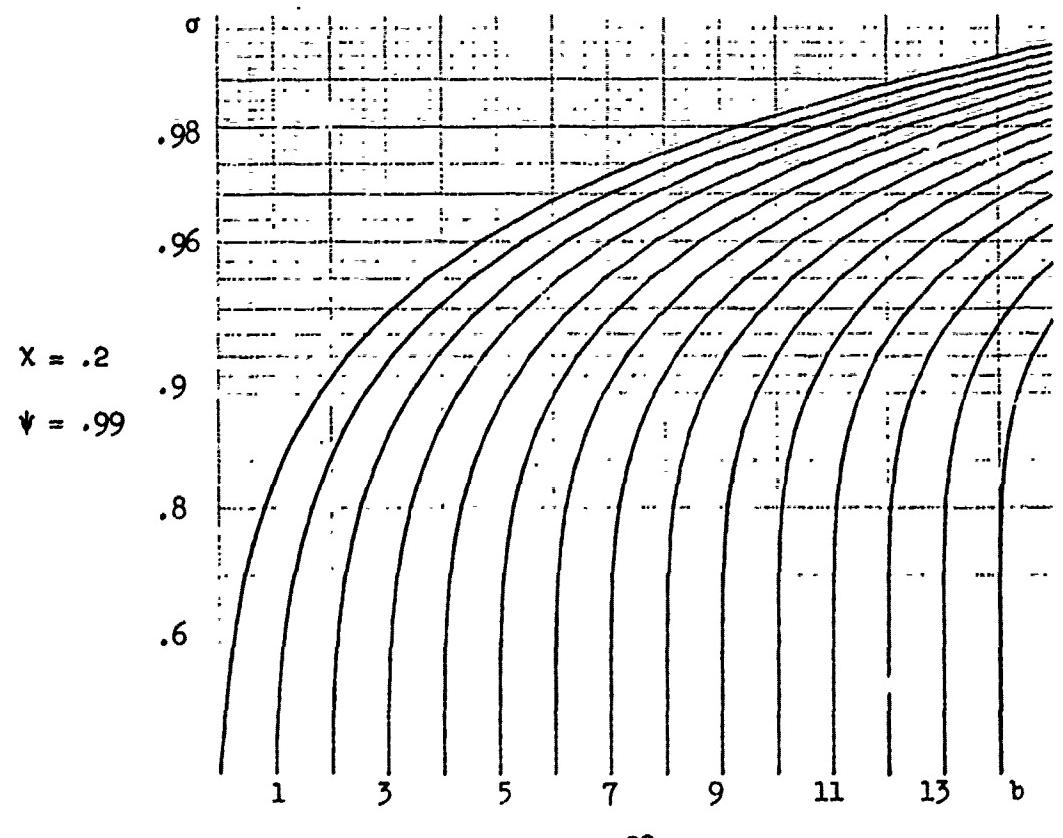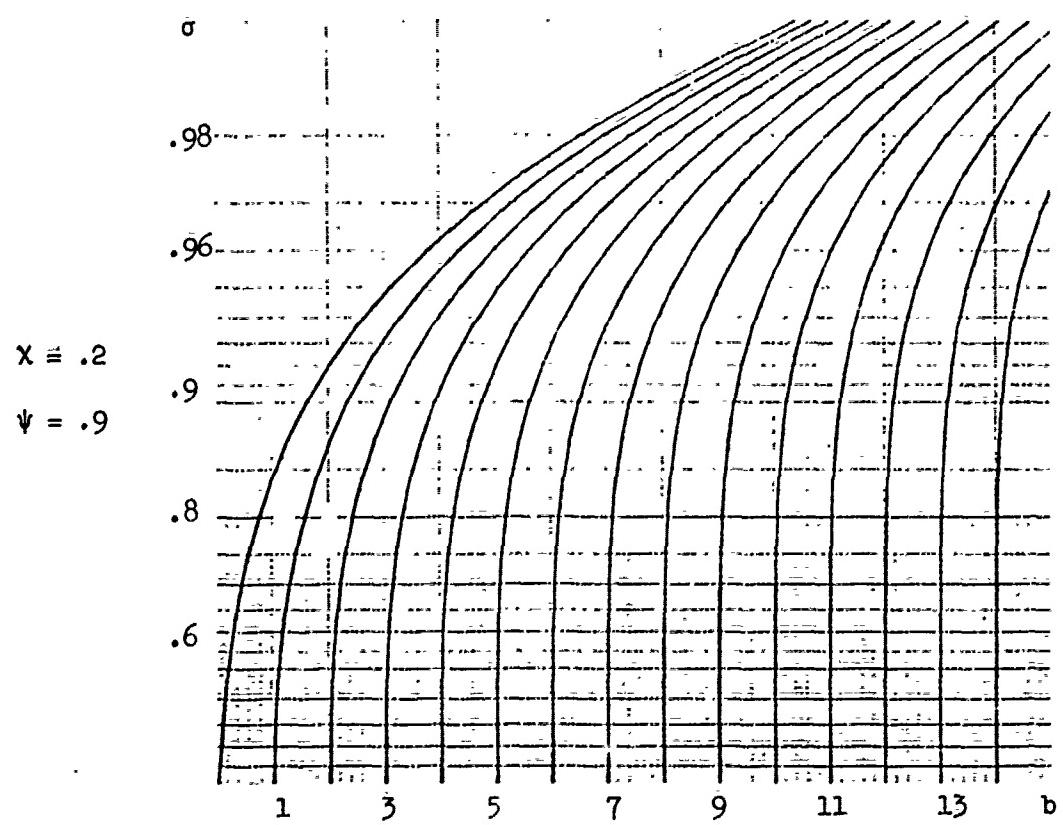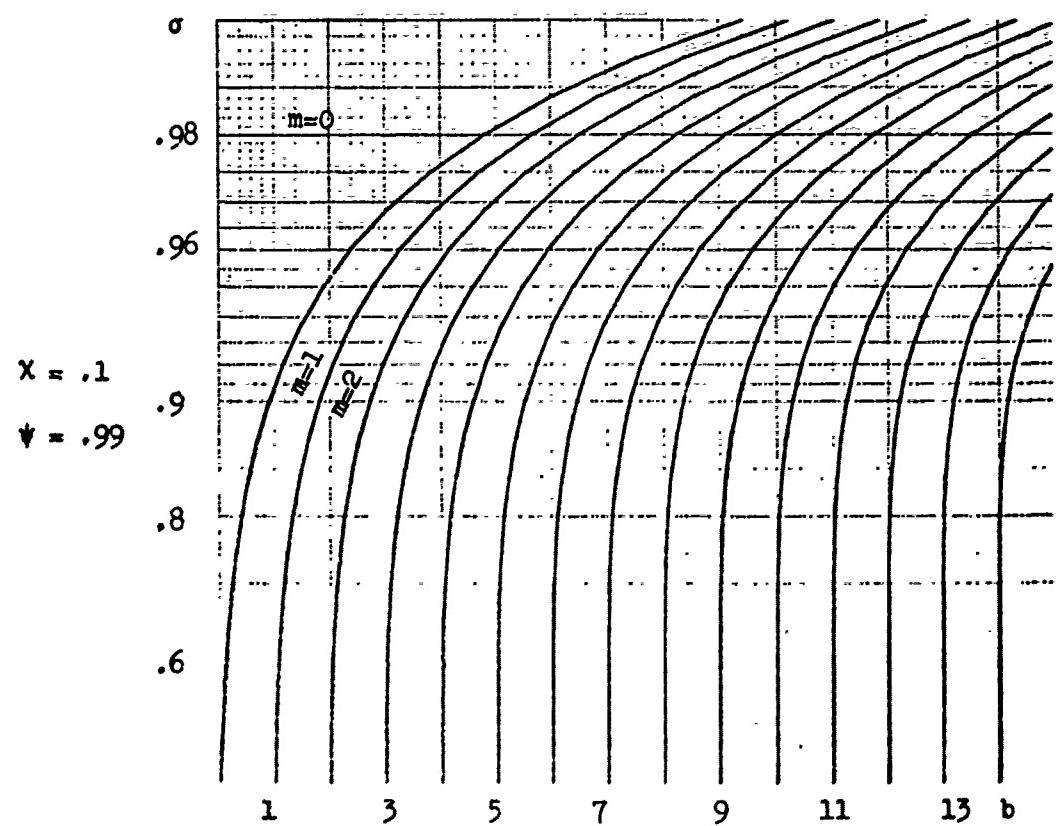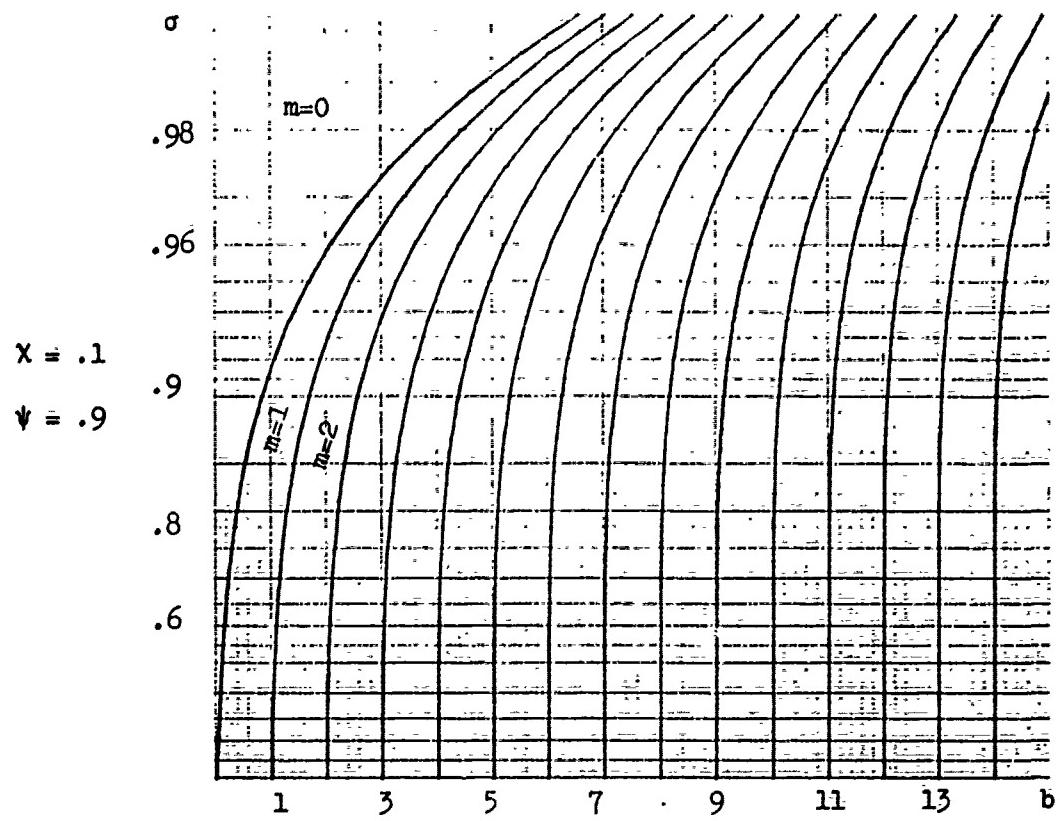
APPENDIX C: The Laplace Transform of the Busy
Period in the $M/E_k/1$ and $M/D/1$
Queueing Systems

This appendix contains graphs showing the Laplace transform of the
busy period in the $M/E_k/1$ and the $M/D/1$ queueing systems. The busy
period is defined as the time from a customer arrives (to an empty system)
until the system becomes empty again. The parameter of the transforms
is denoted by α. As before

$$\sigma \equiv \frac{\lambda}{\lambda + \alpha}$$

and

$$\rho = \frac{\lambda}{\mu} \, ,$$

where λ is the arrival rate and μ is the service rate. The Laplace
transform is denoted by ψ.

For the $M/E_k/1$ queueing systems, ψ is given by

$$\psi = \left(\frac{\frac{k}{\rho}}{\frac{k}{\rho} + \frac{1}{\sigma} - \psi} \right)^{k}, \quad \text{for} \quad k \in N \, .$$

For the $M/D/1$ queueing system, ψ is given by
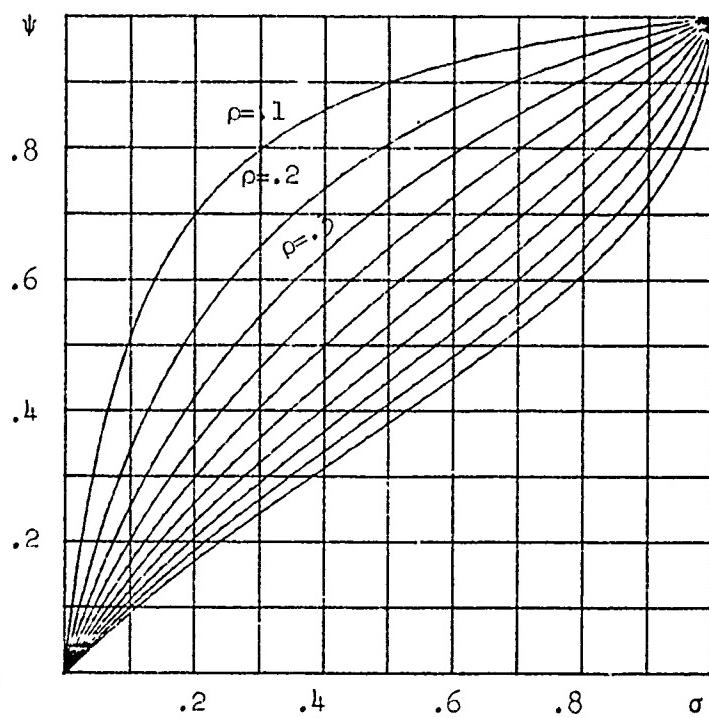
$$\psi = e^{\rho(\psi - \frac{1}{\sigma})} \, .$$

M/E$_1$/1

M/E$_2$/1
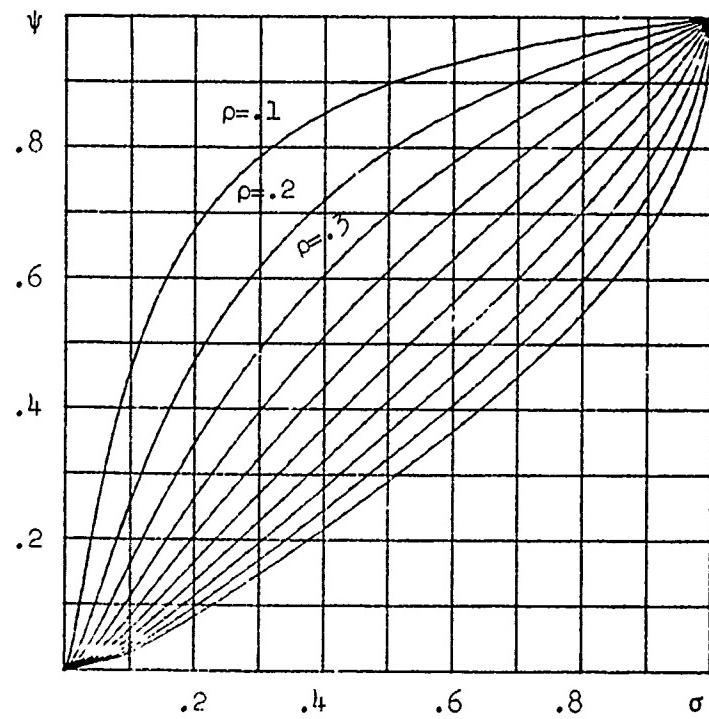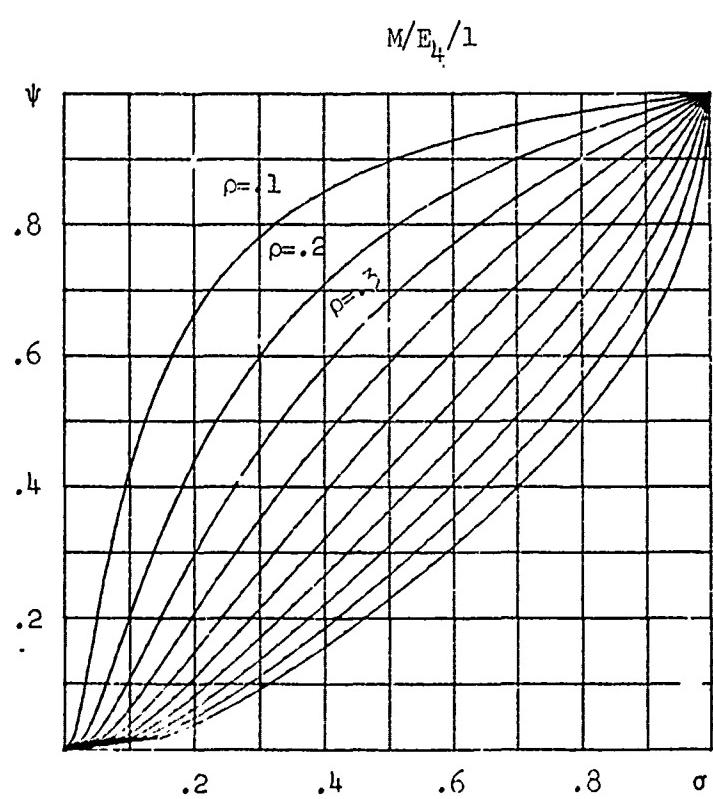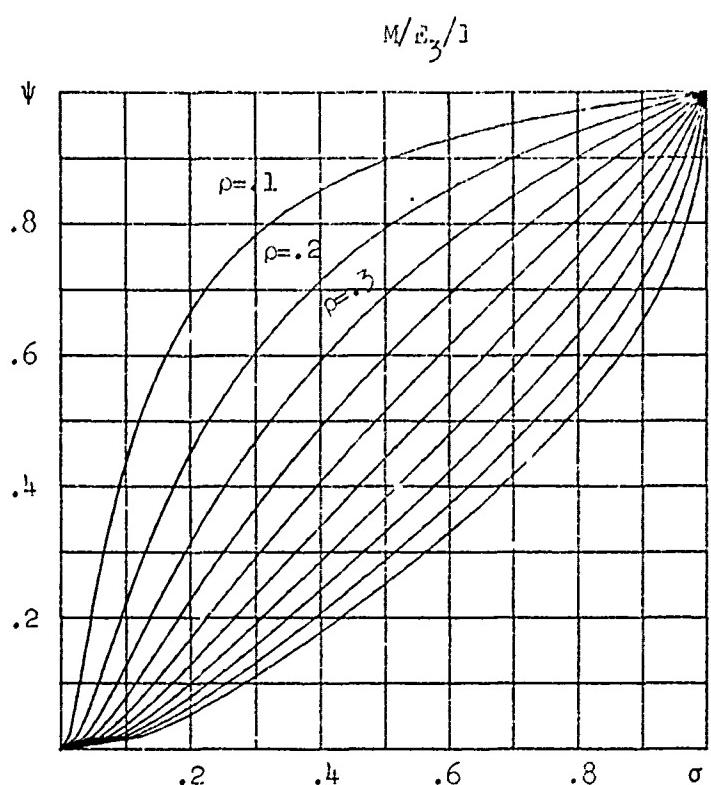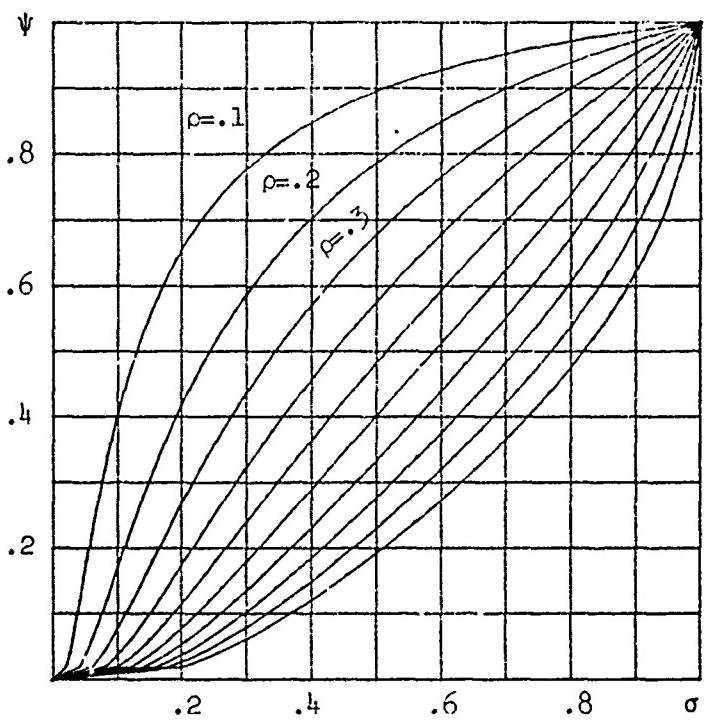
95

$M/E_3/1$

$M/E_4/1$

96

$M/E_9/1$

$M/E_{16}/1$

97

M/E$_2$/1

M/D/1
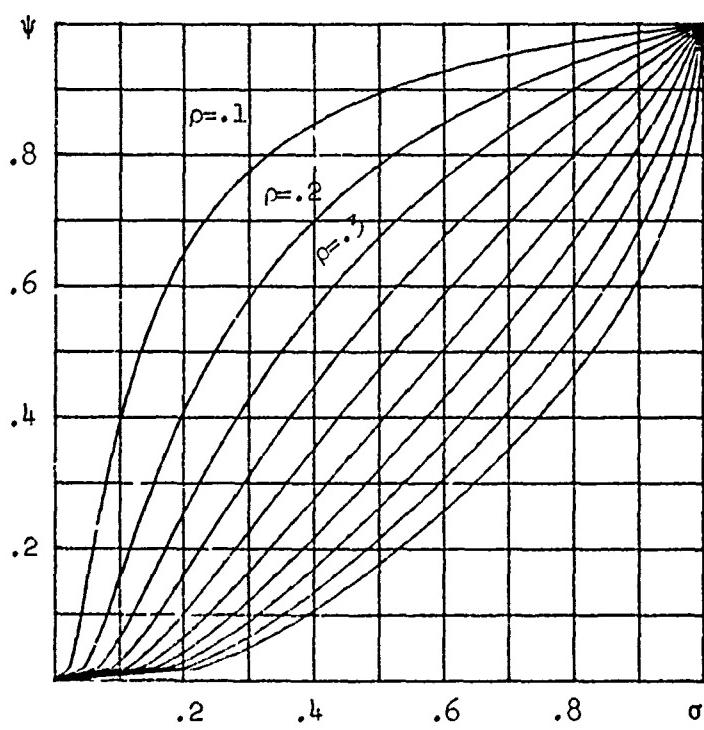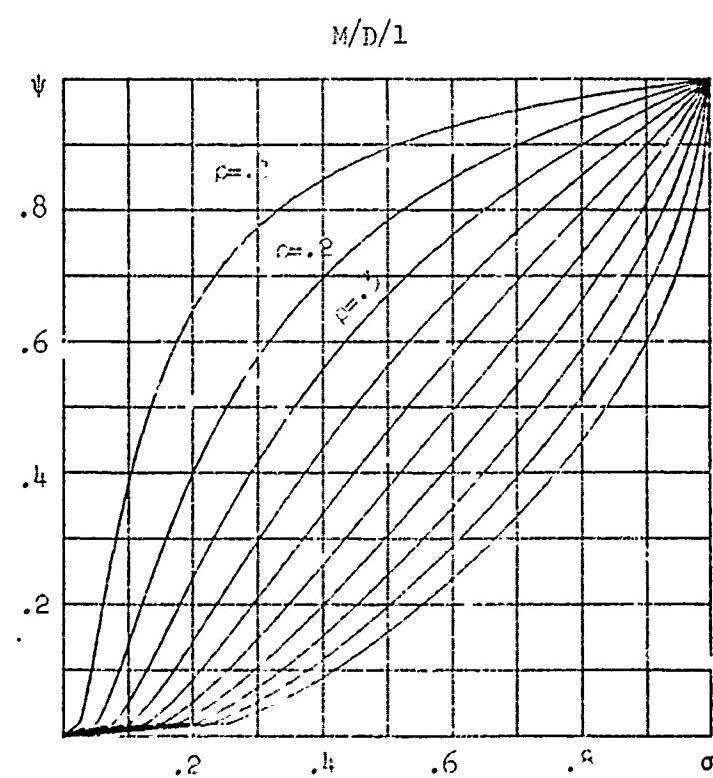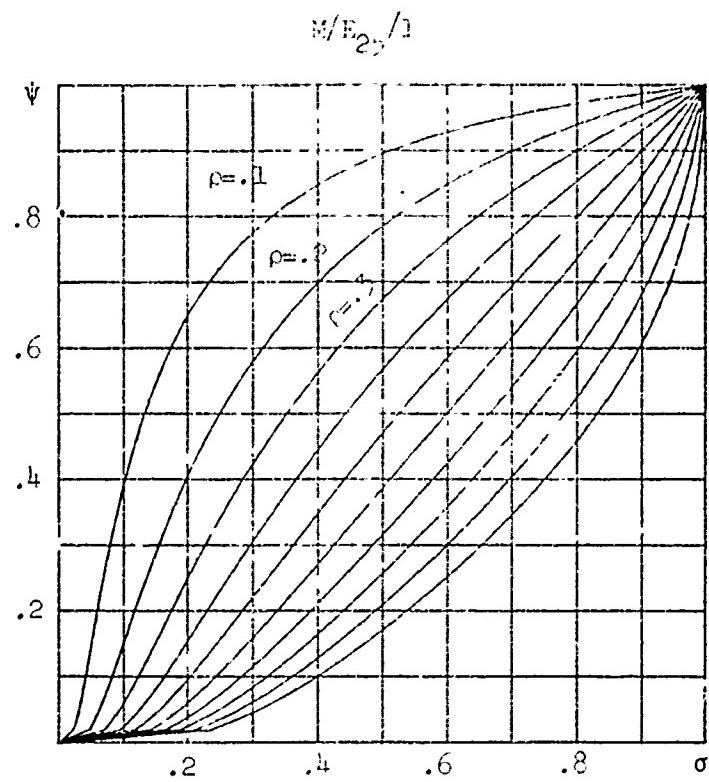
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OPTIMAL CONTROL OF THE M/G/1 QUEUEING SYSTEM WITH REMOVABLE

SERVER-LINEAR AND NON-LINEAR HOLDING COST FUNCTION

by

Peter Orkenyi

Abstract:

This report considers the M/G/1 queueing system with removable server. The cases of linear and non-linear customer holding cost functions are both considered. Non-instantaneous start-up times are allowed. The problem is to find an optimal policy for turning the server on and off. The optimality criteria considered are the average cost criterion, the undiscounted cost criterion and the discounted cost criterion.

A certain class of simple policies, the hysteretic policies, is considered. Natural hysteretic policies and non-degenerate hysteretic policies are introduced. It is shown that there is a natural hysteretic policy which is average optimal, and that if the start-up times are instantaneous or the holding cost function convex, then there is a natural hysteretic policy which is undiscounted optimal. When discounting is used, the results are not as strong, except for the case where the holding cost function is linear. For the non-linear case we still obtain certain fairly weak sufficient conditions for a natural hysteretic policy to be optimal.